Abstract set-theoretic approach to lattice congruences

by Melvin F. Janowitz,
DIMACS, Rutgers University

1 Congruences

Unless otherwise specified, L will denote a finite lattice. This writeup tries to give more details to a discussion that was a part of [10].

Definition 1.1 A *quotient* (denoted s/t) is an ordered pair (s, t) of elements of L with $s \geq t$. Say that $s/t \to u/v$ in one step if for some $w \in L$, $u/v = s \vee w/t \vee w$, or $u/v = s \wedge w/t \wedge w$. Write $s/t \to u/v$ to denote the composition of finitely many relations of the form $x_{i-1}/y_{i-1} \to x_i/y_i$, each in one step, with $x_0/y_0 = s/t$ and the final step ending in $x_n/y_n = u/v$. (Definition from Dilworth [5], p. 349). To say that $s/t \to u/v$ is to say that the quotient s/t is *weakly projective* onto the quotient u/v. Any congruence Θ is completely determined by the quotients it identifies. The reason for this is that $x\Theta y \iff x \vee y \,\Theta\, x \wedge y$.

For any quotient a/b with $a > b$ here is a formula for the smallest congruence Θ_{ab} that identifies a and b. For $x > y$, $x\Theta_{ab}y$ if and only if there exists a finite chain $x = x_0 > x_1 > \cdots > x_n = y$ such that $a/b \to x_{i-1}/x_i$ for $1 \leq i \leq n$. Though we can keep this in mind, there is a much more concise way of looking at all this when we are dealing with finite lattices. We assume unless otherwise specified that L denotes a finite lattice. A *join-irreducible* member of L is an element $j \in L$ such that $j > 0$ and $j > \bigvee\{x \in L : x < j\}$. Thus j has a unique largest element j_* below it. Every element of L is the join of all join-irreducibles below it, so the structure of L is determined by the set $J(L)$ of all join-irreducibles of L. There is a dual notion $M(L)$ of *meet-irreducibles*. Every $m \in M(L)$ is covered by a unique smallest element m^*, and every element of L is the meet of a family of meet-irreducibles. Note that any congruence Θ of L is completely determined by $\{j \in J(L) : j\Theta j_*\}$, so this gives us another way of thinking about congruences. In particular, we can restrict a congruence to $J(L)$, and just worry about whether quotients of the form j/j_* are collapsed. Of course there are dual notions involving meet-irreducibles. We mention the references [3, 4, 6, 7] where some of this is discussed, and briefly present the items we shall need.

Remark 1.2 The material in this remark is taken from Day [4], pp. 398-399, and [3], p. 72.

- For $p, q \in J(L)$, Alan Day [4] writes qCp to indicate that for some $x \in L$, $q \leq x \vee p$ with $q \not\leq x \vee p_*$, thus forcing $q \not\leq x \vee t$ for any $t < p$. Note that for any congruence Θ, if qCp and $p\Theta p_*$, then $q = q \wedge (p \vee x)\Theta q \wedge (p_* \vee x) < q$ forces $q\Theta q_*$. The idea for the C relation is attributed by Day to material from [14]. *Warning:* Some authors write this relation as pDq or qDp.
- A J-set is a subset $J \subseteq J(L)$ such that $p \in J$ with $qCp \Longrightarrow q \in J$.
- **JSet**(L) is the system of all J-sets of L, ordered by set inclusion.

- There is a natural lattice isomorphism between the congruences on L and $(\mathbf{JSet}(L), \subseteq)$. The association is given by mapping the congruence Θ to $J_\Theta = \{j \in J(L) : j\Theta j_*\}$. Going in the other direction, we can construct the congruence associated with a J-set J by using [6], Lemmas 2.33 and 2.34, p. 40 and defining

$$x\Theta_J y \iff \{a \in J(L) : a \le x,\, a \notin J\} = \{a \in J(L) : a \le y,\, a \notin J\}.$$

The ordering of the congruences is given by $\Theta_1 \le \Theta_2 \iff x\Theta_1 y$ implies $x\Theta_2 y$.

- For each $p \in J(L)$, let Φ_p denote the least congruence that makes p congruent to p_*. Then $J_{\Phi_p} = \{q \in J(L) : q\widehat{C}p\}$ where $\widehat{C}$ is the reflexive transitive closure of C. The reader should observe that J_{Φ_p} is the smallest J-set containing p.

- For $p, q \in J(L)$, it is true that $\Phi_q \le \Phi_p \iff q \in \Phi_p \iff q\widehat{C}p$. Thus $\Phi_p = \Phi_q \iff$ both $p\widehat{C}q$ and $q\widehat{C}p$.

We mention that Leclerc and Monjardet were independently led to a similar idea in 1990 (See [11, 13] for a discussion of this). For $p, q \in J(L)$, they write $q\delta p$ to indicate that $q \ne p$, and for some $x \in L$, $q \not\le x$ while $q \le p \vee x$. They show in [11], Lemma 2, that the relations C and δ coincide if and only if L is atomistic. Here an *atom* of a lattice L with 0 is a minimal element of $L \setminus \{0\}$, and L is *atomistic* if every nonzero element of L is the join of a family of atoms. The dual notions of *dual atoms* (coatoms) and *dual atomistic* (coatomistic) are defined in the expected manner.

Definition 1.3 An element s of a lattice L is called *standard* if $(s \vee x) \wedge y = (s \wedge y) \vee (x \wedge y)$ holds $\forall x, y \in L$. Note that the standard elements form a distibutive sublattice of L, and every standard element of L determines a congruence relation Θ_s by the rule $x\Theta_s y$ iff $x \vee y = (x \wedge y) \vee s_1$ for some $s_1 \le s$.

Theorem 1.4 Let L be a finite atomistic lattice. Every congruence relation on L is the minimal one defined by a standard element.

Proof: This is well known and trivial to prove. Yet we supply a proof on the grounds that it builds intuition. Let Θ be a congruence on L. Let $s = \bigvee\{p : p\Theta 0\}$, and note that $s\Theta 0$. Thus for any $x, y \in L$, $(s \vee x) \wedge y\, \Theta\, (0 \wedge x) \wedge y = x \wedge y$. Suppose one could find x, y such that $(s \vee x) \wedge y > (s \wedge y) \vee (x \wedge y)$. There must be an atom p such that $p \le (s \vee x) \wedge y$ but $p \not\le (s \wedge y) \vee (x \wedge y)$. Then $p \not\le x \wedge y$ forces $p\theta 0$, so $p \le s \wedge y$, a contradiction. ∎

2 Results related to relations

Think of an underlying finite lattice L, with $J = J(L)$ the set of join-irreducibles of L. Though we are interested in the congruences of L, it turns out to be useful to abstract the situation, see what can be proved, and then later recapture the deep and natural connection with congruences. This idea was already noted by Grätzer and Wehrung in [8]. The situation serves to illustrate one of the most beautiful aspects of mathematics. Looking at an abstraction of a problem can actually simplify proofs and provide more general results. We ask the reader to bear in mind that though we restrict our attention to finite lattices, we hold open the possibility of establishing a generalization to more general venues. Of course Theorem 1.4 is an example of a result that requires looking at the underlying lattice.

We begin with some notational conventions. Let J be a finite set, and $R \subseteq J \times J$ a binary relation. For $a \in J$, let $R(a) = \{x \in J : aRx\}$, and for $A \subseteq J$, let $R(A) = \bigcup\{R(a) \colon a \in A\}$. The relation R^{-1} is defined by $aR^{-1}b \Leftrightarrow bRa$. A subset V of J is called R-closed if $R(V) \subseteq V$, and R^{-1}-closed if $R^{-1}(V) \subseteq V$. It is easily shown that V is R-closed if and only if its complement $J \setminus V$ is R^{-1}-closed. We are interested in the set $\mathcal{V} = \mathcal{V}_R$ of R^{-1}-closed sets, ordered by set inclusion. We chose R^{-1}-closed sets so as to be consistent with the terminology of Remark 1.2. Clearly $(\mathcal{V}, \subseteq)$ is a sublattice of the power set of J, and has the empty set as its smallest member, and J as its largest member. It will be convenient to simply call any $P \in \mathcal{V}$ a J-set to denote the fact that it is R^{-1}-closed. Note that $P \in \mathcal{V}$ has a complement in $\mathcal{V}$ if and only if $J \setminus P \in \mathcal{V}$. Thus P has a complement if and only if it is both R^{-1}-closed and R-closed.

Remark 2.1 The relation R is said to *reflexive* if jRj for all $j \in J$. It is *transitive* if hRj, jRk together imply that hRk. A relation that is both reflexive and transitive is said to be a *quasiorder*. This is a rather general concept, as every partial order and every equivalence relation is a quasiorder. If the relation R that defines $\mathcal{V}$ is already a quasiorder, then clearly every set of the form $R(a)$ or $R(A)$ is in fact R-closed. Since R^{-1} is also a quasiorder, the same assertion applies to R^{-1}. The relation $R \cap R^{-1}$ is the largest equivalence relation contained in both R and R^{-1}. The least quasiorder containing both R and R^{-1} is denoted $R \vee R^{-1}$, and it is actually also an equivalence relation. The $R \vee R^{-1}$ closed sets are those that are both R and R^{-1} closed.

We could now continue the discussion with a fixed quasiorder R, but we choose instead to have notation that provides an abstract version of Remark 1.2. Accordingly, we take J to be a finite set, but are thinking it as being the join-irreducibles of a finite lattice. A relation R on J is called *irreflexive* if xRx fails for every $x \in J$. We define the relation Δ to be $\{(x, x) : x \in J\}$. We then take R_C to be an irreflexive binary relation on J, and $R_{\widehat{C}}$ the reflexive transitive closure of R_C. By this we mean the transitive closure of $\Delta \cup R_C$. Thus $R_{\widehat{C}}$ is a *quasiorder* of J. Think of qR_Cp as the abstraction of qCp, and $qR_{\widehat{C}}p$ as the abstraction of $q\widehat{C}p$. We are interested in $\mathcal{V} = \{V \subseteq J : p \in V,\, qR_Cp \Longrightarrow q \in V\}$, order it by set inclusion, and call $V \in \mathcal{V}$ a J-set. Note that $\{\emptyset, J\} \subseteq \mathcal{V}$, and that $\mathcal{V}$ is closed under the formation of intersections and unions. Thus $\mathcal{V}$ is a finite distributive lattice. Though R_C is irreflexive, we recall that $R_{\widehat{C}}$ is in fact reflexive by its very construction.

Some intuition may be gleaned from a quick look at what happens when $R_{\widehat{C}}$ is a partial order. We then write $q \leq p$ to denote the fact that $qR_{\widehat{C}}p$. We ask what it means for P to be in $\mathcal{V}$. We note that $p \in P$, $q \leq p$ implies $q \in P$. Thus $\mathcal{V}$ is just the set of order ideals of $(J, \leq)$.

Remark 2.2 Here are some basic facts about $\mathcal{V}$. We remind the reader that each item follows from elementary properties of binary relations; yet each translates to a known property of congruences on a finite lattice.

1. Since $\mathcal{V}$ is closed under the formation of set intersection, it is clear that for each $p \in J$, there is a smallest J-set containing p. We denote it as J_p. Any J-set containing p must clearly also contain $\{q \in V : qR_{\widehat{C}}p\} = R_{\widehat{C}}^{-1}(p)$. Since $\{q \in V : qR_{\widehat{C}}p\}$ is itself a J-set, we deduce that $J_p = \{q \in V : qR_{\widehat{C}}p\}$.

2. Thus $V_p \subseteq V_q \Longleftrightarrow p \in V_q \Longleftrightarrow pR_{\widehat{C}}q$.

3. If $V \in \mathcal{V}$, then $V = \bigcup\{V_p : p \in V\}$. The J-sets of the form V_p are clearly the join-irreducibles of $\mathcal{V}$. To see why, suppose V is not of the form V_p. Then clearly V is not join-irreduciible. On the other hand, if $V = V_p$, let $V' = \bigcup\{V_q \colon V_q \subset V\}$. Evidently $p \notin V'$, so V is join-irreducible.

4. If A is an atom of $\mathcal{V}$, then $p, q \in A \Longrightarrow pR_{\widehat{C}}q$ and $qR_{\widehat{C}}p$, so $(p,q) \in R_{\widehat{C}} \cap R_{\widehat{C}}^{-1}$. Thus $A \in \mathcal{V}$ is an atom iff $A = V_p$ for any $p \in A$.

5. If $R_{\widehat{C}}$ is symmetric, then every join-irreducible is an atom. It follows that $\mathcal{V}$ is atomistic.

6. $R_{\widehat{C}}$ is symmetric if and only if $\mathcal{V}$ is a Boolean algebra.

   Proof: Suppose first that $R_{\widehat{C}}$ is symmetric. We will show that for any $V \in \mathcal{V}$, it is true that $J \setminus V \in \mathcal{V}$. Let $p \in V$ and $q \in J \setminus V$. Suppose rR_Cq. We claim that $r \notin V$. To prove this, we use the symmetry of $R_{\widehat{C}}$ to see that $qR_{\widehat{C}}r$. If $r \in V$, then $qR_{\widehat{C}}r$ would force $q \in V$, contrary to $q \in J \setminus V$, thus showing that $J \setminus V \in \mathcal{V}$. It follows that $\mathcal{V}$ is complemented, so it is a Boolean algebra.

   Suppose conversely that $\mathcal{V}$ is a Boolean algebra. If V_z is an atom of $\mathcal{V}$, then $a \in V_z$ implies $V_a = V_z$, so $a, b \in V_z \Longrightarrow aR_{\widehat{C}}b$. Thus the restriction of $R_{\widehat{C}}$ to V_z is symmetric. What happens if $a \in V_z$ and $b \in J \setminus V_z$? Then both $aR_{\widehat{C}}b$ and $bR_{\widehat{C}}a$ must fail. Since J is the union of all atoms of $\mathcal{V}$ it is immediate that $R_{\widehat{C}}$ is symmetric. ∎

   We note that for congruences on a finite lattice L, this forces the congruence lattice to be a Boolean algebra if and only if the $\widehat{C}$ relation on L is symmetric, thus generalizing many known earlier results that have been established for congruences on lattices.

Remark 2.3 It is well known that associated with every quasiordered set there is a homomorphic image that is a partially ordered set. For the quasiorder $R_{\widehat{C}}$ that we are considering, here is how the construction goes. We say that $p \sim q$ for $p, q \in V$ if $pR_{\widehat{C}}q$ and $qR_{\widehat{C}}p$. Then $\sim$ is an equivalence relation on V, and $\mathcal{V}/\sim$ is a partially ordered set with respect to $\trianglelefteq$ defined by $[p] \trianglelefteq [q]$ if $V_p \subseteq V_q$. One may ultimately show (See Theorem 2.35, p. 41 of [6]) that $(\mathcal{V}, \subseteq)$ is isomorphic to the order ideals of $(\mathcal{V}/\sim, \trianglelefteq)$. If $R_{\widehat{C}}$ is symmetric, then it is an equivalence relation. Though one usually associates with any equivalence relation its associated family of partitions, the set $\mathcal{V}$ of J-sets associated with $R_{\widehat{C}}$ is most certainly rather different.

If $P \in \mathcal{V}$, we want a formula for the pseudo-complement P^* of P. This is the largest member B of $\mathcal{V}$ such that $P \cap B = \emptyset$. A finite distributive lattice is called a *Stone lattice* if the pseudo-complement of each element has a complement.

Theorem 2.4 For $P \in \mathcal{V}$, $P^* = \{q \in J \colon R_{\widehat{C}}^{-1}(q) \cap P = \emptyset\} = J \setminus R_{\widehat{C}}(P)$

Proof: We begin by proving the assertion that $\{q \in J \colon R_{\widehat{C}}^{-1}(q) \cap P = \emptyset\} = J \setminus R_{\widehat{C}}(P)$. This follows from $\{q \in J \colon R_{\widehat{C}}^{-1}(q) \cap P \neq \emptyset\} = R_{\widehat{C}}(P)$. To establish this, note that $q \in R_{\widehat{C}}(P) \Leftrightarrow pR_{\widehat{C}}q$ with $p \in P \Leftrightarrow qR_{\widehat{C}}^{-1}p$ with $p \in P \Leftrightarrow R_{\widehat{C}}^{-1}(q) \cap P \neq \emptyset$. The proof is completed by noting that if $B \in \mathcal{V}$ with $B \cap P = \emptyset$, then $b \in B$, $qR_{\widehat{C}}b \Longrightarrow q \in B$, so $q \notin P$. This shows that $B \subseteq P^*$. ∎

Lemma 2.5 $P \in \mathcal{V}$ has a complement in $\mathcal{V} \iff q \in J \setminus P$, $q_1 R_C\, q$ implies $q_1 \in J \setminus P$.

Proof: The condition is just the assertion that $J \setminus P$ is a J-set. ∎

Theorem 2.6 $\mathcal{V}$ is a Stone lattice if and only if $R_{\widehat{C}}$ has the property that for each $P \in \mathcal{V}$, $q \notin P^*$ implies that either $q \in P$ or else $q \notin P$ and there exists $q_1 \in P$ such that $q_1 R_{\widehat{C}} q$

Proof: This just applies Lemma 2.5 to P^*. ∎

Here is yet another characterization of when $(\mathcal{V}, \subseteq)$ is a Stone lattice. The result for congruences appears in [9], and the proof we present is just a minor reformulation of the proof that was presented therein. We mention an alternate characterization in the spirit of Dilworth's original approach to congruences that was given in [12]. Note that the arguments in [9] were applied to the set of all prime quotients of a finite lattice, where the argument given here applies to any quasiorder defined on a finite set J. We should also mention earlier and stronger results that appear in [15, 16, 17]. So is there anything new in what follows? Only the fact that the proofs can be reformulated for abstract quasiorders.

Theorem 2.7 $\mathcal{V}$ is a Stone lattice if and only if $R_{\widehat{C}}$ has the property that for each $a \in V$ there is one and only one atom V_k of $\mathcal{V}$ such that $V_k \subseteq V_a$.

Proof: Let $P \in \mathcal{V}$, $a \in J$ with V_k the unique atom of $\mathcal{V}$ that is $\subseteq V_a$. Recall that $V_k \subseteq V_a \iff kR_{\widehat{C}}a$.

If $k \notin P$, we let $q \in V$ with $qR_C a$. We will show that $q \notin P$. Let V_j be an atom under V_q. Then $jR_{\widehat{C}}q$, $qR_C a$ forces $jR_{\widehat{C}}a$. Since there is only one atom under a, we must have $V_j = V_k$, so $kR_{\widehat{C}}q$. If $a \in P$, we note that $kR_{\widehat{C}}a$ would put $k \in P$, contrary to $k \notin P$. Thus $a \notin P$. Similarly, $q \in P$ produces a contradiction. Thus $q \notin P$ for any $qR_{\widehat{C}}a$, and this tells us that $a \in P^*$.

If $k \in P$, then $k \in P^{**}$. Replacing P with P^* in the above argument now shows that $a \in P^{**}$. In any case, $a \in J$ implies $a \in P^* \cup P^{**}$ so P^* and P^{**} are complements.

Now assume that for some $a \in V$ there are two atoms V_j and V_k both contained in V_a. If $a \in V_k^{**}$, then $jR_{\widehat{C}}a \Longrightarrow j \in V_k^{**}$. But $V_j \cap V_k = \emptyset$ implies that $V_j \subseteq V_k^*$, a contradiction. If $a \in V_k^*$, then $kR_{\widehat{C}}a$ would put $k \in V_k^*$, contrary to $k \in V_k \subseteq V_k^{**}$. Thus $a \notin V_k^{**} \cup V_k^*$, so V_k^{**} and V_k^* are not complements. ∎

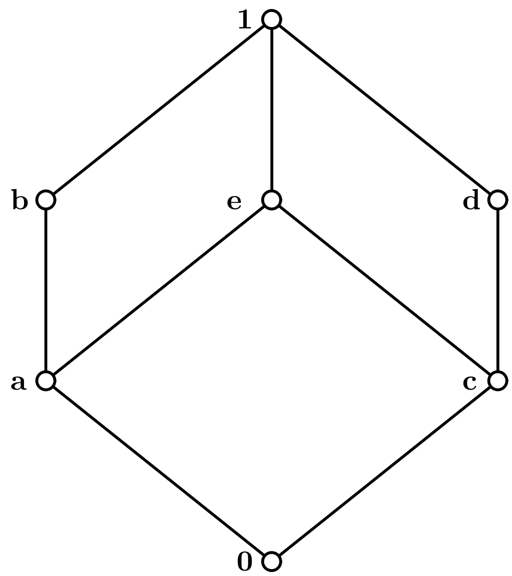

Figure 1: A dual atomistic lattice

Definition 2.8 Let R_C denote an irreflexive binary relation on the finite set J. To say that $\mathcal{V}$ is *subdirectly irreducible* is to say that there is only one atom in $\mathcal{V}$. This is a very old and extremely useful notion in Universal Algebra, and dates back at least to a publication of G. Birkhoff [1]. It negates the idea of a lattice being subdirectly reducible in the sense that the lattice is a sublattice of a nontrivial direct product of lattices. It just states that there is a nontrivial congruence relation that is contained in any other nontrivial congruence.

The following finite version of a result due to S. Radeleczki [15, 16, 17] now pops out.

Corollary 2.9 $\mathcal{V}$ is the direct product of subdirectly irreducible factors if and only if for each $a \in J$ there is only one atom $V_k \subseteq V_a$.

3 An example

Example 3.1 In this example, we let L denote the finite lattice depicted in Figure 1. This lattice was constructed from 2^3 (the Boolean cube) by removal of one atom and all links to that atom. The reader should observe that this lattice is dual atomistic, but not atomistic. The join-irreducibles are a, b, c, d, while the meet-irreducibles are b, d and e. We leave it to the reader to confirm that the C-relation is given by bCa, bCc, bCd, dCa, dCb, dCc, and that the J-sets are

$$\emptyset, \{b,d\}, \{a,b,d\}, \{c,b,d\}, \{a,b,c,d\}.$$

The J-set $\{a,b,d\}$ produces a congruence with two classes $\{a,b,0\}$ and $\{c,d,e,1\}$. By synnetry the classes assoiated with $\{c,b,d\}$ are $\{c,d,0\}$ and $\{a,b,e,1\}$. Finally the classes associated with $\{b,d\}$ are $\{a,b\}$, $\{c,d\}$, $\{e,1\}$, and $\{0\}$. The remaining J-sets lead to trivial congruences.

We now replace L with its dual. This is an atomistic lattice with three atoms b, d and e. To maintain consistent notation, we relabel 0 as $1'$ and 1 as $0'$. The congruence classes for Θ_d are $\{a, b, 1'\}$ and $\{c, d, e, 0'\}$. By symmetry, Θ_e has classes $\{c, d, 1'\}$ and $\{a, b, e, 0'\}$. Finally, Θ_e produces classes $\{a, b\}$, $\{c, d\}$, $\{e, 0'\}$, and $\{1'\}$. We note that the J-sets for the dual of L are

$$\emptyset, \{e\}, \{b, e\}, \{d, e\}\{b, c, d\}.$$

We leave it to the reader to verify these calculations.

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