Some observations on oligarchies, internal direct sums, and lattice congruences

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Abstract

A set-theoretic abstraction of some deep ideas from lattice theory is presented and discussed. By making use of this abstraction, many results from seemingly disparate disciplines can be examined, proved and subtle relationships can be discovered among them. Typical applications might involve decision theory when presented with evidence from sources that yield conflicting optimal advice, insights into the internal structure of a finite lattice, and the nature of homomorphic images of a finite lattice. Some needed historical background is provided.[1] In particular, there is a connection to some early work of Boris Mirkin [25].

1 Background

A new look at some ideas that are related to a pair of landmark results is presented. First among them is Arrow's Theorem [1]. A connection to simple lattices is motivated and discussed in [20]. Secondly, there is John von Neumann's famous construction of a continuous generalization of finite dimensional projective geometries, as presented in his 1936-1937 Princeton lectures (See [28]). These are geometries whose subspaces can have any dimension in the real interval $[0, 1]$. The original definition of a continuous geometry insisted that the underlying lattice be irreducible in the sense that it have no nontrivial direct product decomposition. There was much interest in developing a version that did not have this restriction. This was especially true in light of Kaplansky's famous result [18] that every complete orthocomplemented modular lattice is a continuous geometry. A subdirect sum representation accomplished this in [21, 22], and at a much later date, a topological representation was produced in [12]. Many other authors pondered this question. F. Maeda's work involved the study of a binary relation which we shall denote as $a\nabla b$. It will turn out that failure of this relation has a connection with congruences of an atomistic lattice, and for that reason is useful in connection with the study of simple lattices. We shall expand on this connection in the course of our detailed observations. But first some background material so we are talking the same language. We will assume a basic knowledge of lattice theory, but will quickly establish some needed terminology.

We assume the reader is familiar with partial orders. A *lattice* is a partially ordered set L in which every pair a, b of elements have a *least upper bound* $a \vee b$ and a *greatest lower bound* $a \wedge b$. The smallest member of L will be denoted 0 and its largest element 1. A *bounded* lattice has these distinguished members. Thus for any x in such a lattice, it is true that $0 \leq x \leq 1$. A *congruence relation* on L is an equivalence relation Θ such that $a\Theta b$ implies $a \vee c\,\Theta\, b \vee c$ and $a \wedge c\,\Theta\, b \wedge c$ for all $a, b, c \in L$.

[1]Presented in conjunction with the volume dedicated to the 70th Birthday celebration of Professor Boris Mirkin.

Definition 1.1 A *quotient* (denoted s/t) is an ordered pair (s,t) of elements of L with $s \geq t$. Say that $s/t \to u/v$ in one step if for some $w \in L$, $u/v = s \vee w/t \vee w$, or $u/v = s \wedge w/t \wedge w$. Write $s/t \to u/v$ to denote the composition of finitely many relations of the form $x_{i-1}/y_{i-1} \to x_i/y_i$, each in one step, with $x_0/y_0 = s/t$ and the final step ending in $x_n/y_n = u/v$. (Definition from Dilworth [8], p. 349). To say that $s/t \to u/v$ is to say that the quotient s/t is *weakly projective* onto the quotient u/v. Any congruence Θ is completely determined by the quotients it identifies. The reason for this is that $x \Theta y \iff x \vee y \, \Theta \, x \wedge y$.

For any quotient a/b with $a > b$ here is a formula for the smallest congruence Θ_{ab} that identifies a and b. For $x > y$, $x\Theta_{ab}y$ if and only if there exists a finite chain $x = x_0 > x_1 > \cdots > x_n = y$ such that $a/b \to x_{i-1}/x_i$ for $1 \leq i \leq n$. Though we can keep this in mind, there is a much more concise way of looking at all this when we are dealing with finite lattices. We assume unless otherwise specified that L denotes a finite lattice. A *join-irreducible* member of L is an element $j \in L$ such that $j > 0$ and $j > \bigvee\{x \in L : x < j\}$. Thus j has a unique largest element j_* below it. Every element of L is the join of all join-irreducibles below it, so the structure of L is determined by the set $J(L)$ of all join-irreducibles of L. There is a dual notion $M(L)$ of *meet-irreducibles*. Every $m \in M(L)$ is covered by a unique smallest element m^*, and every element of L is the meet of a family of meet-irreducibles. Note that any congruence Θ of L is completely determined by $\{j \in J(L) : j\Theta j_*\}$, so this gives us another way of thinking about congruences. In particular, we can restrict a congruence to $J(L)$, and just worry about whether quotients of the form j/j_* are collapsed. Of course there are dual notions involving meet-irreducibles. We mention the references [6, 7, 9, 10] where some of this is discussed, and briefly present the items we shall need.

Remark 1.2 The material in this remark is taken from Day [7], pp. 398-399, and [6], p. 72.

- For $p, q \in J(L)$, Alan Day [7] writes qCp to indicate that for some $x \in L$, $q \leq x \vee p$ with $q \not\leq x \vee p_*$, thus forcing $q \not\leq x \vee t$ for any $t < p$. Note that for any congruence Θ, if qCp and $p\Theta p_*$, then $q = q \wedge (p \vee x)\Theta q \wedge (p_* \vee x) < q$ forces $q\Theta q_*$. The idea for the C relation is attributed by Day to material from [29]. *Warning:* Some authors write this relation as pDq or qDp.

- A J-set is a subset $J \subseteq J(L)$ such that $p \in J$ with $qCp \Longrightarrow q \in J$.

- **JSet**(L) is the system of all J-sets of L, ordered by set inclusion.

- There is a natural lattice isomorphism between the congruences on L and $(\mathbf{JSet}(L), \subseteq)$. The association is given by mapping the congruence Θ to $J_\Theta = \{j \in J(L) : j\Theta j_*\}$. Going in the other direction, we can construct the congruence associated with a J-set J by using [9], Lemmas 2.33 and 2.34, p. 40 and defining

  $$x\Theta_J y \iff \{a \in J(L)\colon a \leq x,\, a \notin J\} = \{a \in J(L)\colon a \leq y,\, a \notin J\}.$$

  The ordering of the congruences is given by $\Theta_1 \leq \Theta_2 \iff x\Theta_1 y$ implies $x\Theta_2 y$.

- For each $p \in J(L)$, let Φ_p denote the least congruence that makes p congruent to p_*. Then $J_{\Phi_p} = \{q \in J(L) : q\widehat{C}p\}$ where $\widehat{C}$ is the reflexive transitive closure of C. The reader should observe that J_{Φ_p} is the smallest J-set containing p.

- For $p, q \in J(L)$, it is true that $\Phi_q \leq \Phi_p \iff q \in \Phi_p \iff q\widehat{C}p$. Thus $\Phi_p = \Phi_q \iff$ both $p\widehat{C}q$ and $q\widehat{C}p$.

We mention that Leclerc and Monjardet were independently led to a similar idea in 1990 (See [20, 26] for a discussion of this). For $p, q \in J(L)$, they write $q\delta p$ to indicate that $q \neq p$, and for some $x \in L$, $q \not\leq x$ while $q \leq p \vee x$. They show in [20], Lemma 2, that the relations C and δ coincide if and only if L is atomistic. Here an *atom* of a lattice L with 0 is a minimal element of $L \setminus \{0\}$, and L is *atomistic* if every nonzero element of L is the join of a family of atoms. The dual notions of *dual atoms* (coatoms) and *dual atomistic* (coatomistic) are defined in the expected manner.

2 Results related to relations

Think of an underlying finite lattice L, with $J = J(L)$ the set of join-irreducibles of L. Though we are interested in the congruences of L, it turns out to be useful to abstract the situation, see what can be proved, and then later recapture the deep and natural connection with congruences. This idea was already noted by Grätzer and Wehrung in [11]. The situation serves to illustrate one of the most beautiful aspects of mathematics. Looking at an abstraction of a problem can actually simplify proofs and provide more general results. We ask the reader to bear in mind that though we restrict our attention to finite lattices, we hold open the possibility of establishing a generalization to more general venues.

We begin with some notational conventions. Let J be a finite set, and $R \subseteq J \times J$ a binary relation. For $a \in J$, let $R(a) = \{x \in J : aRx\}$, and for $A \subseteq J$, let $R(A) = \bigcup\{R(a) \colon a \in A\}$. The relation R^{-1} is defined by $aR^{-1}b \Leftrightarrow bRa$. A subset V of J is called R-closed if $R(V) \subseteq V$, and R^{-1}-closed if $R^{-1}(V) \subseteq V$. It is easily shown that V is R-closed if and only if its complement $J \setminus V$ is R^{-1}-closed. We are interested in the set $\mathcal{V} = \mathcal{V}_R$ of R^{-1}-closed sets, ordered by set inclusion. We chose R^{-1}-closed sets so as to be consistent with the terminology of Remark 1.2. Clearly $(\mathcal{V}, \subseteq)$ is a sublattice of the power set of J, and has the empty set as its smallest member, and J as its largest member. It will be convenient to simply call any $P \in \mathcal{V}$ a J-set to denote the fact that it is R^{-1}-closed. Note that $P \in \mathcal{V}$ has a complement in $\mathcal{V}$ if and only if $J \setminus P \in \mathcal{V}$. Thus P has a complement if and only if it is both R^{-1}-closed and R-closed.

Remark 2.1 The relation R is said to *reflexive* if jRj for all $j \in J$. It is *transitive* if hRj, jRk together imply that hRk. A relation that is both reflexive and transitive is said to be a *quasiorder*. This is a rather general concept, as every partial order and every equivalence relation is a quasiorder. If the relation R that defines $\mathcal{V}$ is already a quasiorder, then clearly every set of the form $R(a)$ or $R(A)$ is in fact R-closed. Since R^{-1} is also a quasiorder, the same assertion applies to R^{-1}. The relation $R \cap R^{-1}$ is the largest equivalence relation contained in both R and R^{-1}. The least quasiorder containing both R and R^{-1} is denoted $R \vee R^{-1}$, and it is actually also an equivalence relation. The $R \vee R^{-1}$ closed sets are those that are both R and R^{-1} closed.

We could now continue the discussion with a fixed quasiorder R, but we choose instead to have notation that provides an abstract version of Remark 1.2. Accordingly, we take J to be a finite set, but are thinking it as being the join-irreducibles of a finite lattice. A relation R on J is called *irreflexive* if xRx fails for every $x \in J$. We define the relation Δ to be $\{(x,x) : x \in J\}$. We then take R_C to be an irreflexive binary relation on J, and $R_{\widehat{C}}$ the reflexive transitive closure of R_C. By this we mean the transitive closure of $\Delta \cup R_C$. Thus $R_{\widehat{C}}$ is a *quasiorder* of J. Think of qR_Cp as the abstraction of qCp, and $qR_{\widehat{C}}p$ as the abstraction of $q\widehat{C}p$. We are interested in $\mathcal{V} = \{V \subseteq J : p \in V,\, qR_Cp \Longrightarrow q \in V\}$, order it by set inclusion, and call $V \in \mathcal{V}$ a J-set. Note that $\{\emptyset, J\} \subseteq \mathcal{V}$, and that $\mathcal{V}$ is closed under the formation of intersections and unions. Thus $\mathcal{V}$ is a finite distributive lattice. Though R_C is irreflexive, we recall that $R_{\widehat{C}}$ is in fact reflexive by its very construction.

Some intuition may be gleaned from a quick look at what happens when $R_{\widehat{C}}$ is a partial order. We then write $q \leq p$ to denote the fact that $qR_{\widehat{C}}p$. We ask what it means for P to be in $\mathcal{V}$. We note that $p \in P$, $q \leq p$ implies $q \in P$. Thus $\mathcal{V}$ is just the set of order ideals of $(J, \leq)$.

Remark 2.2 Here are some basic facts about $\mathcal{V}$. We remind the reader that each item follows from elementary properties of binary relations; yet each translates to a known property of congruences on a finite lattice.

1. For each $p \in J$, there is a smallest J-set containing p. We denote this set by V_p, and note that $V_p = \{q \in V : qR_{\widehat{C}}p\} = R_{\widehat{C}}^{-1}(p)$. Thus $V_p \subseteq V_q \Longleftrightarrow p \in V_q \Longleftrightarrow pR_{\widehat{C}}q$. The J-sets V_p are clearly the join-irreducibles of $\mathcal{V}$.

2. If $V \in \mathcal{V}$, then $V = \bigcup\{V_p : p \in V\}$.

3. If A is an atom of $\mathcal{V}$, then $p, q \in A \Longrightarrow pR_{\widehat{C}}q$ and $qR_{\widehat{C}}p$, so $(p,q) \in R_{\widehat{C}} \cap R_{\widehat{C}}^{-1}$. Thus A an atom implies $A = V_p$ for any $p \in A$.

4. $R_{\widehat{C}}$ is symmetric if and only if $\mathcal{V}$ is a Boolean algebra.

   Proof: Suppose first that $R_{\widehat{C}}$ is symmetric. We will show that for any $V \in \mathcal{V}$, it is true that $J \setminus V \in \mathcal{V}$. Let $p \in V$ and $q \in J \setminus V$. Suppose rR_Cq. We claim that $r \notin V$. To prove this, we use the symmetry of $R_{\widehat{C}}$ to see that $qR_{\widehat{C}}r$. If $r \in V$, then $qR_{\widehat{C}}r$ would force $q \in V$, contrary to $q \in J \setminus V$, thus showing that $J \setminus V \in \mathcal{V}$. It follows that $\mathcal{V}$ is complemented, so it is a Boolean algebra.

   Suppose conversely that $\mathcal{V}$ is a Boolean algebra. If V_z is an atom of $\mathcal{V}$, then $a \in V_z$ implies $V_a = V_z$, so $a, b \in V_z \Longrightarrow aR_{\widehat{C}}b$. Thus the restriction of $R_{\widehat{C}}$ to V_z is symmetric. What happens if $a \in V_z$ and $b \in J \setminus V_z$? Then both $aR_{\widehat{C}}b$ and $bR_{\widehat{C}}a$ must fail. Since J is the union of all atoms of $\mathcal{V}$ it is immediate that $R_{\widehat{C}}$ is symmetric. ∎

We note that for congruences on a finite lattice L, this forces the congruence lattice to be a Boolean algebra if and only if the $\widehat{C}$ relation on L is symmetric, thus generalizing many known earlier results that have been established for congruences on lattices.

Remark 2.3 It is well known that associated with every quasiordered set there is a homomorphic image that is a partially ordered set. For the quasiorder $R_{\widehat{C}}$ that we are considering, here is how the construction goes. We say that $p \sim q$ for $p, q \in V$ if $pR_{\widehat{C}}q$ and $qR_{\widehat{C}}p$. Then $\sim$ is an equivalence relation on V, and $\mathcal{V}/\sim$ is a partially ordered set with respect to $\trianglelefteq$ defined by $[p] \trianglelefteq [q]$ if $V_p \subseteq V_q$. One may ultimately show (See Theorem 2.35, p. 41 of [9]) that $(\mathcal{V}, \subseteq)$ is isomorphic to the order ideals of $(\mathcal{V}/\sim, \trianglelefteq)$. If $R_{\widehat{C}}$ is symmetric, then it is an equivalence relation. Though one usually associates with any equivalence relation its associated family of partitions, the set $\mathcal{V}$ of J-sets associated with $R_{\widehat{C}}$ is most certainly rather different.

If $P \in \mathcal{V}$, we want a formula for the pseudo-complement P^* of P. This is the largest member B of $\mathcal{V}$ such that $P \cap B = \emptyset$. A finite distributive lattice is called a *Stone lattice* if the pseudo-complement of each element has a complement.

Theorem 2.4 For $P \in \mathcal{V}$, $P^* = \{q \in J\colon R_{\widehat{C}}^{-1}(q) \cap P = \emptyset\} = J \setminus R_{\widehat{C}}(P)$

Proof: We begin by proving the assertion that $\{q \in J\colon R_{\widehat{C}}^{-1}(q) \cap P = \emptyset\} = J \setminus R_{\widehat{C}}(P)$. This follows from $\{q \in J\colon R_{\widehat{C}}^{-1}(q) \cap P \neq \emptyset\} = R_{\widehat{C}}(P)$. To establish this, note that $q \in R_{\widehat{C}}(P) \Leftrightarrow pR_{\widehat{C}}q$ with $p \in P \Leftrightarrow qR_{\widehat{C}}^{-1}p$ with $p \in P \Leftrightarrow R_{\widehat{C}}^{-1}(q) \cap P \neq \emptyset$. The proof is completed by noting that if $B \in \mathcal{V}$ with $B \cap P = \emptyset$, then $b \in B$, $qR_{\widehat{C}}b \Longrightarrow q \in B$, so $q \notin P$. This shows that $B \subseteq P^*$. ∎

Lemma 2.5 $P \in \mathcal{V}$ has a complement in $\mathcal{V} \iff q \in J \setminus P$, $q_1 R_C\, q$ implies $q_1 \in J \setminus P$.

Proof: The condition is just the assertion that $J \setminus P$ is a J-set. ∎

Theorem 2.6 $\mathcal{V}$ is a Stone lattice if and only if $R_{\widehat{C}}$ has the property that for each $P \in \mathcal{V}$, $q \notin P^*$ implies that either $q \in P$ or else $q \notin P$ and there exists $q_1 \in P$ such that $q_1 R_{\widehat{C}} q$

Proof: This just applies Lemma 2.5 to P^*. ∎

Here is yet another characterization of when $(\mathcal{V}, \subseteq)$ is a Stone lattice. The result for congruences appears in [13], and the proof we present is just a minor reformulation of the proof that was presented therein. We mention an alternate characterization in the spirit of Dilworth's original approach to congruences that was given in [24]. Note that the arguments in [13] were applied to the set of all prime quotients of a finite lattice, where the argument given here applies to any quasiorder defined on a finite set J. We should also mention earlier and stronger results that appear in [30, 31, 32]. So is there anything new in what follows? Only the fact that the proofs can be reformulated for abstract quasiorders.

Theorem 2.7 $\mathcal{V}$ is a Stone lattice if and only if $R_{\widehat{C}}$ has the property that for each $a \in V$ there is one and only one atom V_k of $\mathcal{V}$ such that $V_k \subseteq V_a$.

Proof: Let $P \in \mathcal{V}$, $a \in J$ with V_k the unique atom of $\mathcal{V}$ that is $\subseteq V_a$. Recall that $V_k \subseteq V_a \iff kR_{\widehat{C}}a$.

If $k \notin P$, we let $q \in V$ with qR_Ca. We will show that $q \notin P$. Let V_j be an atom under V_q. Then $jR_{\widehat{C}}q$, qR_Ca forces $jR_{\widehat{C}}a$. Since there is only one atom under a, we must have $V_j = V_k$, so $kR_{\widehat{C}}q$. If $a \in P$, we note that $kR_{\widehat{C}}a$ would put $k \in P$, contrary to $k \notin P$. Thus $a \notin P$. Similarly, $q \in P$ produces a contradiction. Thus $q \notin P$ for any $qR_{\widehat{C}}a$, and this tells us that $a \in P^*$.

If $k \in P$, then $k \in P^{**}$. Replacing P with P^* in the above argument now shows that $a \in P^{**}$. In any case, $a \in J$ implies $a \in P^* \cup P^{**}$ so P^* and P^{**} are complements.

Now assume that for some $a \in V$ there are two atoms V_j and V_k both contained in V_a. If $a \in V_k^{**}$, then $jR_{\widehat{C}}a \Longrightarrow j \in V_k^{**}$. But $V_j \cap V_k = \emptyset$ implies that $V_j \subseteq V_k^*$, a contradiction. If $a \in V_k^*$, then $kR_{\widehat{C}}a$ would put $k \in V_k^*$, contrary to $k \in V_k \subseteq V_k^{**}$. Thus $a \notin V_k^{**} \cup V_k^*$, so V_k^{**} and V_k^* are not complements. ∎

Definition 2.8 Let R_C denote an irreflexive binary relation on the finite set J. To say that $\mathcal{V}$ is *subdirectly irreducible* is to say that there is only one atom in $\mathcal{V}$. This is a very old and extremely useful notion in Universal Algebra, and dates back at least to a publication of G. Birkhoff [2]. It negates the idea of a lattice being subdirectly reducible in the sense that the lattice is a sublattice of a nontrivial direct product of lattices. It just states that there is a nontrivial congruence relation that is contained in any other nontrivial congruence.

The following finite version of a result due to S. Radeleczki [30, 31, 32] now pops out.

Corollary 2.9 $\mathcal{V}$ is the direct product of subdirectly irreducible factors if and only if for each $a \in J$ there is only one atom $V_k \subseteq V_a$.

3 The "del" relation

There is a notion of an internal direct sum of a family of lattices. As a tool toward understanding the internal structure of lattices, there are discussions in [22], pp. 20-25, and [23], pp. 22-24 of what are called internal direct sum decompositions of a lattice with 0. It is shown in both references that the notion of $x\nabla y$ is crucial to this discussion, where $x\nabla y$ indicates that for all $z \in L$, $(x \vee z) \wedge y = z \wedge y$. A more detailed discussion of direct sums occurs in Section 4. As we mentioned earlier, this was motivated by investigations into the structure of continuous geometries. Until recently, the author saw no connection between the ∇ relation and congruences on a finite atomistic lattice. But now let's think of what it means for $p\nabla q$ to fail when p, q are distinct atoms. For some $x \in L$, we must have $(p \vee x) \wedge q > x \wedge q$. Then $q \leq p \vee x$, and $q \not\leq 0 \vee x$. Thus qCp. So the fundamental connection for a finite atomistic lattice is given by the fact that for distinct atoms p, q of such a lattice,

$$p\nabla q \text{ fails} \iff qCp \tag{1}$$

[2] We mention that this is the reason why qR_Cp is taken as the analog of qCp. Having established a connection between the ∇ relation and congruences on a finite atomistic lattice, we look more closely at the del relation on such a lattice. We will restate some pertinent results that were established in [14] back in the 1960s. We mention first that the ∇ relation on arbitrary pairs of elements of a finite atomistic lattice follows quickly from its restriction to pairs of atoms.

Lemma 3.1 In a finite atomistic lattice L, $a\nabla b \iff p\nabla q$ for all atoms $p \leq a$ and $q \leq b$.

Proof: [14], Lemma 6,1, p. 296. ∎

Theorem 3.2 Let L be a finite atomistic lattice. Every congruence relation Θ of L is the minimal congruence generated by an element s that is *standard* in the sense that $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$ for all $r, t \in L$. In fact $x\Theta y \iff (x \vee y) = (x \wedge y) \vee s_1$ for some $s_1 \leq s$.

Proof: Lemma 6.4, p. 297 and Theorem 6.7, p. 298 of [14]. ∎

Theorem 3.3 Let L be a finite dual atomistic lattice. Then $a\nabla b$ in L if and only if $x = (x \vee a) \wedge (x \vee b)$ for all $x \in L$. It follows that $a\nabla b \Longrightarrow b\nabla a$ for all $a, b \in L$.

Proof: Theorem 4.3 of [16]. ∎

Definition 3.4 The element z of a bounded lattice L is called *central* if z has a complement z' such that L is isomorphic to $[0, z] \times [0, z']$ via the mapping $x \mapsto (x \wedge z, x \wedge z')$. There is a discussion of this in [22], p. 37.

Theorem 3.5 Let L be a finite atomistic lattice in which $x\nabla y \Longrightarrow y\nabla x$ for all $x, y \in L$. Then every congruence on L is the congruence generated by a central element of L. Thus the congruences of L form a finite Boolean algebra.

Proof: This follows immediately from a stronger result that appeared in Remark 2.2. Nonetheless, we present a direct lattice theoretic proof. By Theorem 3.2, every congruence on L is the minimal one generated by a standard element s. If q is an atom disjoint from s, then $s\nabla q$. By symmetry of ∇, $q\nabla s$. It is immediate that if $t = \bigvee\{\text{ atoms } q \in L\colon q \not\leq s\}$, then $t\nabla s$. Thus s and t are complements. For any $x \in L$, we note that $x = (x \wedge s) \vee (x \wedge t)$. For if this failed there would be an atom $r \leq x$ such that $r \not\leq (x \wedge s) \vee (x \wedge t)$. But then $r \wedge s = r \wedge t = 0$, a contradiction. Thus s is central ([23], Theorem 4.13, p. 18). ∎

Corollary 3.6 Every finite atomistic lattice in which ∇ is symmetric is a direct product of simple lattices. In particular, this is true for any finite lattice that is both atomistic and dual atomistic.

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[2] Evidently this was known to B. Monjardet and N. Caspard as early as 1995 (Monjardet, private communication).

Here a lattice is called *simple* if it admits no non-trivial congruence. It follows immediately from Remark 2.2 that finite simple lattices are characterized by the fact that for every pair j, k distinct join-irreducibles, $j\widehat{C}k$. A distributive lattice is simple if and only if it has at most 2 members. One might wonder why Corollary 3.6 leads to a direct product of simple lattices while Proposition 7.2 of [32] leads to a direct product of subdirectly irreducible lattices. The reason is that in the finite case, every congruence relation is the minimal one generated by a central element of the lattice.

It would be interesting to further investigate generalizations of the del relation that are valid for finite lattices that are not atomistic. We outline the start of such a project. For elements a, b of a finite lattice L, we write $a\diamondsuit b$ to denote the fact that they are not comparable (in symbols $a \parallel b$) and for all $x \in L$, $(x \vee a) \wedge b = [x \vee (a \wedge b)] \wedge b$. Note that if $a \wedge b = 0$, this just says that $a\nabla b$. The reason for assuming $a \parallel b$ is that otherwise the assertion that $(x \vee a) \wedge b = [x \vee (a \wedge b)] \wedge b$ is trivially true. In order to obtain a form of separation axiom along the lines of aCb and $a\delta b$, it is convenient to write $a\zeta b$ to indicate that a, b are join-irreducibles with $a \parallel b$ such that $a \leq b \vee x$ and $a \not\leq (a \wedge b) \vee x$ for some $x \in L$. Note that $aCb \Longrightarrow a \not\leq b$, and $a\zeta b \Longrightarrow a \parallel b$. For $a \parallel b$, it is evident that $aCb \Longrightarrow a\zeta b \Longrightarrow a\delta b$. We might mention that an obvious modification of the proof of [20], Lemma 2 will establish that $\delta = \zeta \iff L$ is atomistic. It is interesting to note that by the same Lemma, $\zeta = C$ if and only if L is atomistic. This follows from the fact that if $x < j$ for any join irreducible j, there must then exist a join-irreducible j' with $j' \leq x < j$. Though we have defined ζ and δ to be relations on $J(L)$, it is true that both relations make sense for any elements of L. We begin our discussion of the diamond relation with a generalization of Theorem 3.3. This result relates equational identities with conditions that involve implications that involve inequalities.

Theorem 3.7 Let L be a dual atomistic finite lattice. For $a, b \in L$ with $a \parallel b$, the following are equivalent:

(1) $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b)$ for all $x \in L$.

(2) $a\diamondsuit b$.

(3) $b \leq a \vee x \Longrightarrow b \leq (a \wedge b) \vee x$ for all $x \in L$.

(4) $b \leq a \vee d \Longrightarrow b \leq (a \wedge b) \vee d$ for all dual atoms d of L.

Proof: (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) is obvious, and true for all finite lattices.

(4) $\Longrightarrow$ (1) Suppose (4) holds and $x \vee (a \wedge b) < (x \vee a) \wedge (x \vee b)$. Using the fact that L is dual atomistic, there must exist a dual atom $d \geq x \vee (a \wedge b)$ such that $d \vee [(x \vee a) \wedge (x \vee b] = 1$. Then $d \vee a = d \vee b = 1$. But now $b \leq d \vee a \Longrightarrow b \leq (a \wedge b) \vee d = d$, contrary to $b \vee d = 1$. ∎

Corollary 3.8 Let L be a finite dual atomistic lattice. If $a, b \in L$, then $a\diamondsuit b \iff$ for every dual atom d it is true that $a \wedge b \leq d \Longrightarrow a \leq d$ or $b \leq d$.

Proof: By applying the Theorem with $x = d$ any dual atom, we see that $a \wedge b \leq d \Longrightarrow a \leq d$ or $b \leq d$. Suppose conversely that the condition holds. For arbitrary $x \in L$, we choose d as in the proof of (4) $\Longrightarrow$ (1) of Theorem 3.7, and apply the condition. ∎

Corollary 3.9 In any dual atomistic finite lattice $a \Diamond b$ implies $b \Diamond a$.

Proof: We apply the Theorem to $a \Diamond b$, and note that if $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b)$ for all $x \in L$, then $b \Diamond a$. ∎

Remark 3.10 Let L be a finite dual atomistic lattice with a, b non-comparable join-irreducibles. Evidently $a \wedge b \leq a_*$ and $a \wedge b \leq b_*$. Suppose $a \Diamond b$. Let $x \in L$ be fixed but arbitrary. Using the fact that $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b)$, we see that

$(x \vee a) \wedge (x \vee b) \leq x \vee a_*$,

$(x \vee a) \wedge (x \vee b) \leq x \vee b_*$, so

$(x \vee a) \wedge (x \vee b) \leq (x \vee a_*) \wedge (x \vee b_*)$.

It is immediate that

$(x \vee a) \wedge (x \vee b) = (x \vee a_*) \wedge (x \vee b_*)$, and so

$(x \vee a) \wedge (x \vee b) = (x \vee a_*) \wedge (x \vee b) = (x \vee a) \wedge (x \vee b_*)$.

Thus $a \wedge (x \vee b) = a \wedge (x \vee b_*)$ and $b \wedge (x \vee a) = b \wedge (x \vee a_*)$.

This shows that $a \leq (x \vee b) \Longrightarrow a \leq (x \vee b_*)$ and $b \leq (x \vee a) \Longrightarrow b \leq (x \vee a_*)$, so both aCb and bCa will fail. Thus for a, b non-comparable join-irreducibles of a finite dual atomistic lattice,

$$a \Diamond b \Longrightarrow aCb \text{ and } bCa \text{ must both fail.} \tag{2}$$

Example 3.11 We present an example to illustrate the approach to congruences on a finite lattice via J-sets. Let L be the five element non-modular lattice N_5 with coverings $0 < a < b < 1$ and $0 < c < 1$. The join-irreducibles are then a, b, c with $a_* = c_* = 0$ and $b_* = a$. This example is discussed on p. 38 of [10]. There are five J-sets: $\emptyset$, $\{b\}$, $\{a, b\}$, $\{b, c\}$, $\{a, b, c\}$. The J-set $\{b\}$ only produces a single merger of $\{a, b\}$, while the J-set $\{a, b\}$ has two classes $\{0, a, b\}$ and $\{c, 1\}$. Finally, the J-set $\{b, c\}$ has the two mergers $\{a, b, 1\}$ and $\{0, c\}$. Note the connection with the fact that L is isomorphic with its dual.

Example 3.12 We next have an example that illustrates what can go wrong for a finite lattice that is not dual atomistic. Let $L = \{0, a, b, c, d, 1\}$ with coverings $0 < a < b < 1$ and $0 < c < d < 1$. The join-irreducibles are $\{a, b, c, d\}$ with $a_* = c_* = 0$, $b_* = a$ and $d_* = c$. Note that $\{b\}$ is a J-set since the merger of b with a is a lattice congruence. Note though that $d \leq b \vee c$, $d \leq b_* \vee c$, and $d \not\leq (d \wedge b) \vee c = c$. Thus $d \zeta b$ does not force d to be a member of the J-set $\{b\}$.

We mention the obvious fact that every result involving finite dual atomistic lattices has a corresponding dual result that is true for finite atomistic lattices.

Example 3.13 *In this example, we let L denote the finite lattice depicted in Figure 1. This lattice was constructed from 2^3 (the Boolean cube) by removal of one atom and all links to that atom. The reader should observe that this lattice is dual atomistic, but not atomistic. The join-irreducibles are a, b, c, d, while the meet-irreducibles are b, d and e. We leave it to the reader to confirm that the C-relation is given by bCa, bCc, bCd, dCa, dCb, dCc, and that the J-sets are*

$$\emptyset, \{b, d\}, \{a, b, d\}, \{c, b, d\}, \{a, b, c, d\}.$$

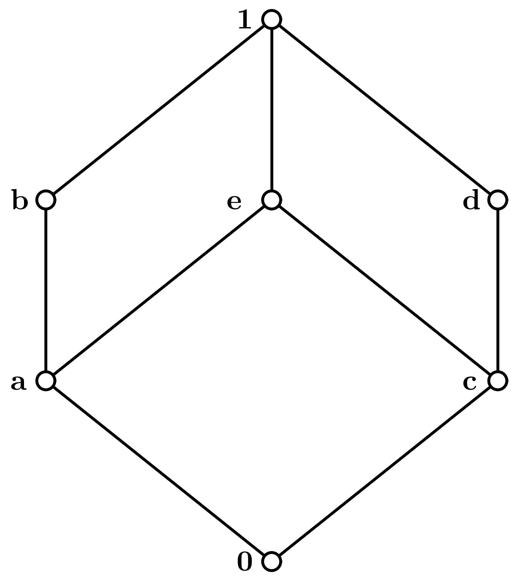

Figure 1: A dual atomistic lattice

We now ask what it means for $a\Diamond b$ to fail for a, b distinct non-comparable join-irreducibles on a finite dual atomistic lattice L. By Theorem 3.7, this is equivalent to the existence of a dual atom d for which $b \leq a \vee d$ with $b \not\leq (a \wedge b) \vee d$. Thus failure of $a\Diamond b$ is equivalent to $b\zeta a$. If follows that the ζ-relation is symmetric. For a finite atomistic lattice, this should be compared to failure of $a\nabla b$ being equivalent to bCa. Note the connection with Corollary 15, p. 502 of [26].

4 Internal Direct Sums of a Finite Lattice

Let $S_1, S_2, \ldots, S_n$ be subsets of a lattice L with 0. Following the terminology of F. Maeda [21], we say that L is the internal direct sum of the S_i if

(1) Each $x \in L$ may be written as $x = \bigvee_{1 \leq i \leq n} x_i$ with $x_i \in S_i$, and

(2) $x \in S_i$, $y \in S_j$ with $i \neq j$ forces $x\nabla y$.

Each S_i is called a *direct summand* of L. There is also a notion of an external direct product of the S_i given by taking the direct product of the family $\{S_i : 1 \leq i \leq n\}$ with the partial order $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n) \iff a_i \leq b_i \, \forall i$. There is then a natural isomorphism between the external direct product of the family S_i and its internal direct sum. It is given by $(a_1, a_2, \ldots, a_n) \longleftrightarrow \bigvee_i a_i$ (See pp. 21-22 of [22]). Having said this, we plan to simplify our notation and identify these two isomorphic entities.

The key item for thinking about all this appears as Theorem 1, p. 1 of [15]. This characterizes direct summands of any lattice L with 0 as central elements of the lattice of ideals of L. For a finite lattice, every ideal is principal, so this tells us that direct summands are generated by the central elements of the lattice. Here is the connection with ∇. By

[23], Theorem 4.13, p. 18, in any bounded lattice L, z central in L is equivalent to the existence of an element z' such that $z \nabla z'$, $z' \nabla z$, and $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in L$. The connection with the $\diamondsuit$ relation comes from the fact that in any bounded lattice L,

$$a \diamondsuit b \iff a \nabla b \text{ in } [a \wedge b, 1]. \tag{3}$$

If z is central in L, then clearly $z \vee a$ is central in $[a, 1]$. It would be interesting to investigate the structure of finite lattices where every central member of any filter $[a, 1]$ is of this form. The dual of this condition has been studied for many years, and is called the *relative center property* (RCP). This condition was studied in [17] and examples as well as references were provided therein. The reader might also consult [5] where a connection is given between RCP and congruences in orthomodular lattices. Meaningful examples of what we are discussing may be obtained by just looking at the dual of any lattice that satisfies the relative center property. This leads us to investigate the structure of $\{x \in L\colon x \diamondsuit c\}$ in a finite lattice L. We present a partial result. Further investigation is called for.

Lemma 4.1 Let a, b, c be elements of the finite lattice L. Then $a \diamondsuit c, b \diamondsuit c \Longrightarrow (a \vee b) \diamondsuit c$.

Proof: Note first that by applying the definition of $\diamondsuit$ twice, we have

$(*)$ $(a \vee b) \wedge c = [(a \wedge c) \vee b] \wedge c = [(b \wedge c) \vee (a \wedge c)] \wedge c = (a \wedge c) \vee (b \wedge c)$.

Then for any $x \in L$, and again making two uses of the definition of $\diamondsuit$, followed by an application of $(*)$, we write

$$\begin{aligned}[(a \vee b) \vee x)] \wedge c &= [a \vee (b \vee x)] \wedge c = [(a \wedge c) \vee (b \vee x)] \wedge c \\ &= [b \vee (a \wedge c) \vee x] \wedge c = [(b \wedge c) \vee x \vee (a \wedge c)] \wedge c \\ &= [(a \wedge c) \vee (b \wedge c) \vee x] \wedge c = [[(a \vee b) \wedge c] \vee x] \wedge c\,. \ \blacksquare\end{aligned}$$

Remark 4.2 We mention that any orthomodular as well as any complemented modular lattice that satisfies RCP has the stronger property that the center of any proper interval $[a, b]$ consists of the set of all $(z \vee a) \wedge b$ with z central in L. This is proved using the natural isomorphism of $[a, b]$ with an interval of the form $[0, c]$. It would be interesting if this could be extended to a larger class of relatively complemented lattices. We also mention Theorem 4.4 of [17] where it is shown that for a complete orthomodular lattice RCP is equivalent to e central in $[0, e \vee f]$ with $e \wedge f = 0$ implies $e \nabla f$.

We turn now to a deeper consideration of the structure of a finite atomistic lattice L in which the ∇ relation is symmetric. Recall that for each atom a of L, the smallest J-set containing a is given by $J_a = \{q \in J(L)\colon q \widehat{C} a\}$. We note that J_a generates the smallest congruence relation Θ_a for which a is congruent to 0. By Theorem 3.4, this is the congruence generated by the central element $e(a)$, which is the smallest central element above a. By Theorem 2.6, the pseudocomplement of J_a is given by $J_a^* = J \setminus R_{\widehat{C}}(a)$.

In what follows a, b are distinct atoms of L. Since $e(a), e(b)$ are atoms of the center of L, there are only two possibilities: either $e(a) = e(b)$, or $e(a) \wedge e(b) = 0$. For the atoms a, b there are three possibilities: bCa, or bCa fails but $b\widehat{C}a$, or $b\widehat{C}a$ fails. Recall from equation (1) that $a \nabla b$ fails $\iff$ bCa.

Lemma 4.3 $b\widehat{C}a \Longrightarrow e(a) = e(b)$.

Proof: Recall that $e(a), e(b)$ are atoms of the center of L. Suppose bCa and that $e(a) \wedge e(b) = 0$. We know that there is an $x \in L$ such that $b \leq a \vee x$ and $b \not\leq x$. Then

$$b = b \wedge e(b) \leq e(b) \wedge (a \vee x) = (e(b) \wedge a) \vee (e(b) \wedge x) \leq x,$$

a contradiction. Since $\widehat{C}$ is the transitive closure of C, it follows that $e(a) = e(b)$, and this completes the proof. ∎

Lemma 4.4 Suppose $b\widehat{C}a$ fails and $q \in J(L)$ with qCb. Then $q\widehat{C}a$ fails. It follows that $b \in J_a^*$, so $e(a) \wedge e(b) = 0$.

Proof: If $q\widehat{C}a$, then by symmetry of ∇, bCq with $q\widehat{C}a$ forces $b\widehat{C}a$, a contradiction. ∎

Theorem 4.5 Let L be a finite atomistic lattice in which the ∇ relation is symmetric. Then L is either a Boolean lattice, or it is simple with $a\widehat{C}b$ for all pairs of atoms a, b or it is a direct sum of such lattices.

Definition 4.6 In a bounded lattice L, a pair of elements a, b is said to be *perpsective* if there is an element x such that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x = 0$. The symbolism for this is $a \sim b$. The transitive closure for a perspective to b is called a *projective* to b.

Lemma 4.7 Let L be finite, atomistic and dual atomistic. If a, b are distinct atoms of L, failure of $a\nabla b$ is equivalent to $a \sim b$. Hence $a \sim b \iff bCa$. This is true also for the dual of L.

Proof: Suppose $a\nabla b$ fails. There must exist an $x \in L$ such that $x < (x \vee a) \wedge (x \vee b)$. Choose a dual atom $t \geq x$ such that $t \not\geq (x \vee a) \wedge (x \vee b)$. Then $t \not\geq a$ and $t \not\geq b$, so $t \vee a = t \vee b = 1$. Since a, b are atoms, we have $t \wedge a = t \wedge b = 0$. Thus $a \sim b$. The converse is obvious. ∎

Theorem 4.8 Every finite atomistic and dual atomistic lattice is either a Boolean lattice or is a simple lattice in which any pair of atoms is projective and in the relation $\widehat{C}$ and dually for dual atoms, or is the internal direct sum of such lattices. In particular this is true for any finite relatively complemented lattice.

Remark 4.9 We would be remiss if we did not at least mention the connection between a direct summand of a finite lattice and the results from Section 2. If we let R_C denote an irreflexive relation on the finite set J, we recall that the J-set P is a direct summand of $\mathcal{V}$ if and only if $J \setminus P$ is an J-set. See Lemma 2.5.

5 Oligarchies

This entire manuscript has as its original inspiration the appearence of the recent paper [4] by Chambers and Miller. Here there is presented a lattice theoretic characterization of when a decision algorithm is an *oligarchy*. An improved result due to Leclerc and Monjardet appears in [20]. The earliest reference the author could find where a lattice theoretic background is provided for a consensus of partitions is the one provided by Boris Mirkin in [25]. This was refined in [19]. See also [27]. We shall be working in a finite lattice L. Intuition may be provided by thinking of L as a model for describing the behavior of a partition of society, or of a partial order or of some concrete decision problem. We shall follow the notation of [20], but will briefly mention here the relevant terminology and notation.

Remark 5.1 A *consensus algorithm* is a mapping $F\colon L^n \to L$, where L^n is the product of $N = n$ copies of L. We agree to let π denote a typical *profile* $\pi = (x_1, x_2, \ldots, x_n)$ of members of L^n, and $N_x(\pi) = \{i \in N\colon x \leq x_i\}$. To say that F is *Paretian* is to say that for any atom a, if $N_a(\pi) = N$, then $a \leq F(\pi)$. To say that F is *decisive* is to say that if $N_a(\pi) = N_a(\pi')$ then $a \leq F(\pi) \Leftrightarrow a \leq F(\pi')$. F is *neutral monotone* if for all atoms a, a', and all profiles π, π', $N_a(\pi) \subseteq N_{a'}(\pi')$ implies that if $a \leq F(\pi)$ then $a' \leq F(\pi')$. The constant function that sends every profile π to 0 is denoted F^0.

Finally to say that F is an *oligarchy* is to say that there is a subset M of the indexing set N such that for every profile π, $F(\pi) = \bigwedge\{\pi_i\colon i \in M\}$. For $x \in L$, we agree to let $\pi_x = (x, x, \ldots, x)$ denote the constant profile having each component x. A mapping $F\colon L_n \to L$ is called *residual* if it is a meet homomorphism such that $F(\pi_1) = 1$. We mention Theorem 5 of [20] in which the following conditions are shown to be equivalent for any finite simple atomistic lattice L having cardinality greater than 2 and any consensus function $F\colon L^n \to L$.

Theorem 5.2 The following conditions are equivalent:

1. F is decisive and Paretian.

2. F is neutral monotone and is not F^0.

3. F is a meet homomorphism and $F(\pi) \geq \bigwedge \pi$ for all profiles π.

4. F is a residual map and $F(\pi_a) \geq a$ for every atom a.

5. F is an oligarchy.

We pause to provide a bit of intuitive motivation for the subject at hand. Suppose for the moment that you are in charge of production quotas for a large manufacturing company and that you have an advisory committee consisting of n agents. Each agent i gives you advice in the form of a partition x_i of the space of all possible actions D you might take, and on the basis of these n partitions for π, you must decide on an action $F(\pi)$. The partitions of D may be viewed as a finite simple lattice that is both atomistic and dual atomistic, so we are in a setting where Theorem 5.2 may be applied. Further motivation is provided in [4]. This makes an interesting connection between properties of social choice functions and pure lattice theoretic ideas. It would be interesting to see if this result could be extended to a somewhat broader class of lattice. The key observation is in Corollary 3.6. Making use of this result, we may move from results on a finite simple lattice to results on a direct product of finite simple lattices. Thus we have a characterization of oligarchies on any atomistic finite lattice in which the ∇ relation is symmetric, and in particular for any finite lattice which is both atomistic and dual atomistic. Here specifically is what we have in mind. Let $L_1, L_2, \ldots, L_k$ each denote finite simple lattices having cardinality > 1, and in which the ∇ relation is symmetric. Let F_i be an oligarchy on L_i for each i. Let $L = \Pi_i L_i$ and let F be defined on L by $F(\pi)$ having its ith component the output of F_i applied to the restriction of π to L_i. Then F is a form of generalized oligarchy. It would be of interest to extend Theorem 5.2 to this situation.

6 An epilogue

We close by reviewing the natural tie between the abstract relation theoretic approach in Section 2 and the deep results developed by a number of authors. We especially mention Alan Day [6, 7], and the book by Freese, Ježek and Nation [9].

Remark 6.1 Here then are the main ideas that were covered for the study of congruences on a finite lattice L.

(a) Failure of the ∇ relation on an atomistic lattice and its connection with the C relation. This is discussed in Section 3. See equation (1).

(b) The C relation on an arbitrary finite lattice and its abstraction to an irreflexive relation R_C defined on a finite set V.
This is Section 2. Remark 2.1 and Theorems 2.4 and 2.6. The abstract formulation can be used to find a generalization of conditions that guarantee that the congruences form a Boolean algebra or a Stone lattice. Noting that $\widehat{C}$ is always symmetric for any finite simple lattice, it might be interesting to have an example of a finite simple lattice in which the C relation is not symmetric. It would also be of interest to apply the results more generally to other finite quasiordered sets.

(c) In Section 3, a generalization of the ∇-relation was introduced and denoted as $a\Diamond b$. There are now three types of separation conditions under consideration. Further work on the connection between these conditions might be appropriate.

Underlying Equation $\forall x \in L$	Symbol	Separation Condition for some $x \in L$
$(x \vee b) \wedge a = (x \vee b_*) \wedge a$	aCb	$a \leq b \vee x$ and $a \not\leq b_* \vee x$; $a \not\leq b$
$(x \vee b) \wedge a = (x \vee (a \wedge b)) \wedge a$	$a\zeta b$	$a \leq b \vee x$ and $a \leq (a \wedge b) \vee x$; $a \parallel b$
$(x \vee b) \wedge a = x \wedge a$	$a\delta b$	$a \leq b \vee x$ and $a \not\leq x$; $a \neq b$

(d) Section 4 considers internal direct sums of a finite lattice, and further explores the connection between the relations ∇ and $\diamondsuit$. As an application, the structure of certain finite atomistic lattices are discussed.

(e) Section 5 gave a quick look at a recent lattice theoretic connection with oligarchies.

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References

[1] Arrow, K., *Social Choice and Indvidual Welfare*, Wiley, New York, 1972.

[2] Birkhoff, G., *Subdurect unions in universal algebra*, Bulletin Amer. Math. Soc. **50**, 1944, 764–768.

[3] Blyth, T. S., and Janowitz, M. F., *Residuation Theory*, Pergamon, 1972.

[4] Chambers, C. P. and Miller, A. D., *Rules for aggregating information*, Social Choice and Welfare **36**, 2011, 75–82.

[5] Chevalier, G. *Around the relative center property in orthomodular lattices*, Proc. American Math. Society **112**, 1991, 935–948.

[6] Day, A., *Characterization of finite lattices that are bounded homomorphic images or sublattices of free lattices*, Can. J. Math. **31**, 1978, 69-78.

[7] Day, A., *Congruence normality: The characterization of the doubling class of convex sets*, Algebra Universalsis **31**, 1994, 397–406.

[8] Dilworth, R. P., *The structure of relatively complemented lattices*, Annals of Math., Second Series **51**, 1950, 348–359.

[9] Freese, R., Ježek, J., and Nation, J. B., *Free Lattices*, Mathematical Surveys and Monographs **42**, American Mathematical Society, 1991.

[10] Grätzer, George, *The Congruences of a Finite Lattice, A Proof-by-Picture Approach*, Birkhäuser, Boston, 2006.

[11] Grätzer, G. and Wehrung, F., *On the number of join-irreducibles in a congruence representation of a finite distributive lattice*, Algebra Universalis **49** (2), 165-178.

[12] Harding, J. and Janowitz, M. F. *A bundle representation representation for continuous geometries*, Advances in Applied Math. **19**, 1997, 282–293.

[13] Iqbalunisa, *On lattices whose lattices of congruence relations are Stone lattices*, Fund. Math. **70**, 1971, 315–318.

[14] Janowitz, M. F., *A characterization of standard ideals*, Acta Math. Acad. Sci. Hung. **16**, 1965, 289–301.

[15] Janowitz, M. F., *A note on normal ideals*, Journal of Science Hiroshima University, Series A-I, **30**, 1966, 1–8.

[16] Janowitz, M. F., *Section semicomplemented lattices*, Math. Z. **108**, 1968, 63–76.

[17] Janowitz, M. F., *Separation conditions in relatively complemented lattices*, Colloq. Math. **22**, 1970, 25–34.

[18] Kaplansky, I., *Any orthocomplemented complete modular lattice is a continuous geometry*, Annals of Math., Second Series **61**, 1955, 524-541.

[19] Leclerc, B., *Efficient and binary consensus function on transitively defined relations*, Math. Social Sciences **8**, 1984, 45–61.

[20] Leclerc, B. and Monjardet, B., *Aggregation and Residuation*, Order **30**, 2013, 261–268.

[21] Maeda, F., *Direct sums and normal ideals of lattices*, Journal of Science Hiroshima University, Series A **14**, 1949, 85–92. q

[22] Maeda, F., *Kontinuierlichen Geometrien*, Springer-Verlag, 1958.

[23] Maeda, F. and Maeda, S., *Theory of symmetric lattices*, Springer-Verlag, 1970.

[24] Malliah, C. and Bhatta, P. S., *On lattices whose congruences form Stone lattices*, Acta Math. Hung. **49**, 1987, 385–389.

[25] Mirkin, B. *On the problem of reconciling partitions* In: Quantitative Sociology, International Perspectives on Mathematical and Statistical Modelling, 441-449, Academic Press, New York, 1975.

[26] Monjardet, B. and Caspard, N., *On a dependence relation in finite lattices*, Discrete Mathematics **165/166**, 1997, 497–505.

[27] Nehring, K., *Oligarchies in judgement aggregation*. Working paper

[28] von Neumann, J. *Continuous Geometry*, Princeton, 1960 and 1998.

[29] Pudlâk, P. and Tuma, J., *Yeast graphs and fermentation of algebraic lattices*, Colloq. Math, Soc. János Bolyai; Lattice Theory, Szeged, 1971, 301–341, North-Holland, Amsterdam.

[30] Radeleczki, S., *Some structure theorems for atomistic algebraic lattices*, Acta Math. Hung. **86**, 2000, 1–15.

[31] Radeleczki, S., *Maeda-type decomposition of CJ-generated algebraic lattices*, Southeast Asian Bulletin of Mathematics **25**, 2001, 503–513.

[32] Radeleczki, S., *The direct decomposition of l-algebras into products of subdurectly irreducible factors*, Journal Aust. Math. Soc. **75**, 2003, 41–56.