Generalized Oligarchies

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Abstract

Recent events at the Boston Marathon demonstrate a need for a group of individuals to quickly and simultaneously make a number of important decisions. This leads to a study of the direct product of oligarchies each having the same collection of agents, but analyzing different issues.

1 Background

Remark 1.1 Let n be a fixed positive integer, and $N = \{1, 2, \ldots, n\}$. For the finite lattice L, let L^n denote the direct product of n copies of L. We agree to let π denote a typical *profile* $\pi = (x_1, x_2, \ldots, x_n)$ of members of L^n, and $N_x(\pi) = \{i \in N : x \le x_i\}$. The constant profile π_x is defined by $\pi_x = (x, x, \ldots, x)$. A *consensus function on* L is a mapping $F : L^n \to L$. To say that the consensus function F is *Paretian* is to say that for any $a \in L$, if $N_a(\pi) = N$, then $a \le F(\pi)$. To say that F is *decisive* is to say that if $N_a(\pi) = N_a(\pi')$ then $a \le F(\pi) \Leftrightarrow a \le F(\pi')$. F is *neutral monotone* if for all $a, a' \in L$, and all profiles π, π', $N_a(\pi) \subseteq N_{a'}(\pi')$ implies that if $a \le F(\pi)$ then $a' \le F(\pi')$.

Finally to say that F is an *oligarchy* is to say that there is a subset M of the indexing set N such that for every profile π, $F(\pi) = \bigwedge\{\pi(j) : j \in M\}$. A mapping $F : L^n \to L$ is called *residual* if it is a meet homomorphism such that $F(\pi_1) = 1$. The mapping $F^0 : L^n \to L$ is defined by $F^0(\pi) = 0$ for every profile π. We begin by restating a Theorem of Leclerc and Monjardet. This result generalizes a result from [2] and is of great interest because it provides a link between ideas that originated in the realm of social choice theory with results coming from the structure theory of partially ordered sets.

Theorem 1.2 ([4],Theorem 5) Let L be a finite simple atomistic lattice having cardinality greater than 2, and $F : L^n \to L$ a consensus function on L. The following conditions are then equivalent:

(F1) F is decisive and Paretian.

(F2) F is neutral monotone and is not F^0.

(F3) F is a meet homomorphism and $F(\pi) \ge \bigwedge_j \pi(j)$ for all profiles π.

(F4) F is a residual map and $F(\pi_a) \ge a$ for every atom a.

(F5) F is an oligarchy.

Definition 1.3 For a finite atomistic lattice L, Monjardet ([5], p. 51) introduces a relation δ on pairs of atoms by letting $a\delta b$ denote the fact that $a \ne b$ and for some $x \in L$, $a < b \vee x$ with $a, b \not\le x$

Remark 1.4 Though the proof given by Monjardet and Leclerc in [4] of (F4) $\Longrightarrow$ (F5) is very clear, we restate it here to aid the reader's intuition. We begin with the idea of a residuated map. We agree to let L, M denote finite lattices, with $F: L \to M$ and $G: M \to L$ mappings. To say that F is *residual* is to say that F is a meet homomorphism such that $F(1) = 1$. Recall that G is *residuated* if G is a join homomorphiem such that $G(0) = 0$. Every residual map $F: L \to M$ has a unique residuated mapping $G: M \to L$ associated with it. The two are each isotone and are related by the equation

$$G(y) \leq x \iff y \leq F(x).$$

The mapping G may be directly defined from F by taking $G(y) = \bigwedge\{x \in L: y \leq F(x)\}$. Properties of residuated and residual mappings are developed in detail in [1]. We now repeat the proof by Leclerc and Monjardet of Theorem 1.2, (4) $\Longrightarrow$ (5). Recall that L is a finite simple atomistic lattice having cardinality at least 3, and that $F: L^n \to L$ is a residual map such that for every atom a of L, $F(\pi_a) \geq a$. We need to know how F gets to be an oligarchy. Let $G: L \to L^n$ be the residuated map associated with F. Apply the isotone mapping G to the ineqality $a \leq F(\pi_a)$ to obtain $G(a) \leq GF(\pi_a) \leq \pi_a$. Thus for any index i, $G_i(a) \in \{0, a\}$. Here G_i is the ith component of the mapping G. Let $M(a) = \{i \in N: G_i(a) = a\}$.

Claim 1: For distinct atoms a and b, $a\delta b \Longrightarrow M(a) \subseteq M(b)$.

Proof: Since $a\delta b$ there is an $x \in L$ such that $a < b \vee x$ and $a \not\leq x$. Since L is atomistic, there is a finite family of atoms K such that $a \leq \bigvee K$, $a \notin K$, while $b \in K$. We may clearly assume K is such a family having minimal cardinality. Then $a \leq \bigvee K$. Applying the residuated mapping G to this inequality produces $G(a) \leq G(\bigvee K) = \bigvee\{G(c): c \in K\}$. By minimality of the family K, $G_i(a) = a \Longrightarrow G_i(c) = c\ \forall c \in K$. It is immediate that $M(a) \subseteq M(c)\ \forall c \in K$. ∎

Clain 2: If L is simple, then $M(a) = M(b)$ for all atoms a, b.

Proof: For L simple there is a path involving δ from a to b, and another path from b to a. ∎

Now if $M = M(a)$ for any atom a, then $a \leq F(\pi) \Longleftrightarrow G(a) \leq GF(\pi) \leq \pi$. Hence for each coordinate $i \in M$, $a = G_i(a) \leq \pi(i)$, so $a \leq \pi(i)$ for all $i \in M$.

Thus $a \leq F(\pi) \iff a \leq \pi(i)\, \forall i \in M \iff a \leq \bigwedge\{\pi(i): i \in M\}$. Since L is atomistic, it follows that $F(\pi) = \bigwedge\{\pi(i): i \in M\}$. ∎

We pause to provide a bit of intuitive motivation for the subject at hand. Suppose for the moment that you are in charge of production quotas for a large manufacturing company and that you have an advisory committee consisting of n agents. Each agent i gives you advice in the form of a partition x_i of the space of all possible actions D you might take, and on the basis of these n partitions produced by π, you must decide on one or more actions $F(\pi)$. The partitions of D may be viewed as a finite simple lattice that is both atomistic and dual atomistic, so we are in a setting where Theorem 1.2 may be applied. Further motivation is provided in [2]. This makes an important connection between properties of social choice functions and pure lattice theoretic ideas. It would be interesting to see if this result could be extended to a somewhat broader class of lattice. The key observation is in Corollary 3.2 of [3], which states that every finite atomistic lattice in which the ∇ relation is symmetric is necessarily a direct product of simple lattices. Making use of this result, we may move from results on a finite simple lattice to results on a direct product of finite simple lattices. Thus we have results for any atomistic finite lattice in which the ∇ relation is symmetric, and in particular for any finite lattice which is both atomistic and dual atomistic. Here $a\nabla b$ means that $(a \vee x) \wedge b = x \wedge b$ for all $x \in L$. For a, b atoms there is a natural connection with the δ relation given by $a\nabla b \iff b\delta a$ fails (See [3]). Here specifically is what we have in mind. Let $L_1, L_2, \ldots, L_k$ each denote finite simple lattices having cardinality > 2, and assume $F_i\colon L_i^n \to L_i$ is an oligarchy on L_i for each i. Let $L = \prod_i L_i$ and let F be defined on L^n by $F(\pi)$ having its ith component the output of F_i applied to the restriction of π to L_i. Alternately, we could take L to be the internal direct sum of the L_i lattices, and then view $F(\pi)$ as the union of the outputs of the F_i outputs. Either way, F is a form of generalized oligarchy. It would be of interest to extend Theorem 1.2 to this situation.

2 The axioms for a generalized oligarchy

There will be some duplication of notation between this discussion and the comments made in Section 1. Assume L is a finite atomistic ∇-symmetric lattice that is the internal direct sum of k simple lattices, each having more than 2 members. Let $z_1, z_2, \ldots, z_k$ denote the k atoms of $Z(L)$, the center of L. Then each interval $[0, z_i]$ is a simple lattice having at least 3 members, and every atom of L is a member of exactly one interval of the form $[0, z_i]$. Let n be a positive integer, and L^n the direct product of n copies of L. Let $F\colon L^n \to L$ be a consensus function. We would like to relate properties of F to properties of a family of induced consensus functions $(F_i\colon 1 \le i \le k)$, where $F_i\colon [0, z_i]^n \to [0, z_i]$. For each profile $\pi = (x_1, x_2, \ldots, x_n)$ of members of L^n, let $\pi_i = \pi_{z_i} \wedge \pi = (x_1 \wedge z_i, x_2 \wedge z_i, \ldots, x_n \wedge z_i)$ and $F_i(\pi_i)$ to be $F(\pi) \wedge z_i$. Please note that if π, π' are profiles, then $\pi_i = \pi'_i$ for all i implies that $\pi = \pi'$. We would like also to know that if $\pi_i = \pi'_i$ for a specific index i, then $F_i(\pi) = F(\pi')$. In other words, we need to know that F is *summand compatible* in the sense that $\pi_i = \pi'_i \Longrightarrow F(\pi) \wedge z_i = F(\pi') \wedge z_i$ for all profiles π and π'. This assumption allows us to lift properties of a direct summnd of L to corresponding properties of L. We must make sure that each F_i is well defined. Let $\pi^\imath$ be a profile on $[0, z_i]$, and let π^* denote $\pi^\imath$ viewed as a profile on L. Then $(\pi^*)_i = \pi^\imath$ so $F_i(\pi^\imath)$ has been defined. Note further that for any profile π of L^n, $\pi_i = (\pi_i)_i$, so $F(\pi) \wedge z_i = F(\pi_i) \wedge z_i$.

At this point there are two sets of indexing symbols under consideration. To help clarify the notation, we agree to use the subscript i when referring to one of the simple lattices $[0, z_i]$; with the subscript j reserved to specify a specific member of a profile.

Theorem 2.1 Let F be a summand compatible consensus function on a finite atomistic lattice L, where L is a direct sum of simple lattices, each of which has at least three members. The following conditions are then equivalent:

(P1) F is decisive and Paretian.

(P2) F is neutral monotone but is not F^0.

(P3) F is a meet homomorphism and $F(\pi) \ge \bigwedge_j \pi(j)$ for any profile π.

(P4) F is a residual map and $F(\pi_a) \ge a$ for every atom a.

(P5) F is a *generalized oligarchy* in the sense that for every atom z_i of the center of L, the induced consensus funnction F_i defined on $[0, z_i]$ by $F_i(\pi \wedge \pi_{z_i}) = F(\pi) \wedge z_i$ is an oligarchy.

Proof: Recall that for some integer $n > 1$, $F\colon L^n \to L$, and $F_i\colon [0, z_i]^n \to [0, z_i]$. We will attempt to relate each property (Pk) to the corresponding property (Fk) from Theorem 1.2, and then use the results of that Theorem. The arguments rest on the fact that for any atom a of L, if $a \leq z_i$, then for any profile π of L^n, it is true that

(1) $$a \leq F_i(\pi_i) \iff a \leq F(\pi) \wedge z_i \iff a \leq F(\pi).$$

Suppose first that each F_i is Paretian. For $a \in L$, recall that π_a denotes the constant profile $(a, a, \ldots, a)$, and note that if $a \leq z_i$, then $a \leq F_i(\pi_a)_i$, so $a \leq F(\pi_a)$. This shows that F is Paretian. The converse implication F Paretian implies each F_i is Paretian is clear, so we have that F is Paretian if and only if every F_i is Paretian. We turn now to the connection between F being decisive and each F_i being decisive. For the atom a of L, we note that $a \leq z_i$ in $L \iff a \leq z_i$ in each factor of L^n. Thus

(2) $$N_a(\pi) \text{ (in } L) = N_a(\pi_i) \text{ (in } [0, z_i])$$

The connection between F and each F_i being decisive is now clear.

We turn next to property (P2). Let F satisfy (P2), Recall from equation (2) that for any atom $a \in L$, if $a \in [0, z_i]$, then $N_a(\pi) = N_a(\pi_i)$. It follows easily that F neutral monotone is equivalent to every F_i being neutral monotone. The fact that $F \neq F^0$ is equivalent to every F_i not being F^0 is obvious.

Suppose next that F is a meet homomorphism. Let π, π' be profiles. Then for any index $i \leq k$, $F_i(\pi \wedge \pi')_i = F_i(\pi \wedge \pi' \wedge \pi_{z_i}) = F(\pi \wedge \pi' \wedge \pi_{z_i}) \wedge z_i = F(\pi \wedge \pi_{z_i}) \wedge z_i \wedge F(\pi' \wedge \pi_{z_i}) \wedge z_i = F_i(\pi_i) \wedge F_i(\pi_i')$. This shows that F a meet homomorphism implies each F_i is a meet homomorphism. Suppose that $F(\pi) \geq \bigwedge_j \pi(j)$. Then $F(\pi) \wedge z_i \geq \bigwedge_j (\pi(j) \wedge z_i)$, so $F_i(\pi_i) \geq \bigwedge_j (\pi_i(j))$. Thus F satisfying (P3) implies every F_i satisfies (P3). Now suppose every F_i satisfies (P3). Then for each $i \leq k$

$$F(\pi \wedge \pi') \wedge z_i = F_i(\pi \wedge z_i \wedge \pi' \wedge z_i) = F_i(\pi \wedge z_i) \wedge F_i(\pi' \wedge z_i) = F(\pi) \wedge z_i \wedge F(\pi') \wedge z_i.$$

Since L is the direct sum of the intervals $[0, z_i]$, it follows that F is a meet homomorphism. Noting that $F(\pi) \geq \bigwedge_j \pi(j) \iff F_i(\pi_i) \geq \bigwedge_j \pi_i(j)$, we see that F satisfies (P3) if and only if every F_i satisfies (F3).

At this point we may apply Theorem 1.2 and deduce the equivalence of (P1), (P2), and (P3). Noting that (P3) trivially implies (P4) for F, and that (P4) for F forces (F4) for each induced function F_i, a second application of Theorem 1.2 completes the proof. ∎

Open issues: The nature of the summand compatibility condition needs investigation. We also wonder what happens for finite atomistic lattices that are not ∇-symmetric, or what happens when L is not atomistic, or for other types of consensus functions. Here is what can now be said when ∇ is symmetric on an atomistic lattice.

Notation: Unless otherwise specified, we now assume that L is a finite atomistic lattice and that it is isomorphic to a finite direct product of simple lattices each of which has cardinality at least three. We also assume that $F\colon L^n \to L$ is a consensus function, and that $z_1, z_2, \ldots, z_k$ denote the atoms of the center of L, so each interval $[0, z_i]$ is a simple atomistic lattice.

Lemma 2.2 If $F(\pi_{z_i}) = z_i$ for every atom z_i of the center of L, theb F is summand compatible.

Proof: $F_i(\pi \wedge \pi_{z_i}) = F(\pi) \wedge z_i = F(\pi) \wedge F(\pi_{z_i})$. ∎

Lemma 2.3 If $F(\pi \wedge \pi_{z_i}) = F(\pi) \wedge F(\pi_{z_i})$ and $F(\pi_{z_i}) \geq z_i$ for all i, then F is summand compatible.

Proof: Let $F(\pi_{z_i}) = t$, where $t \geq z_i$. Then $F(\pi \wedge \pi_{z_i}) = F(\pi) \wedge t$, so if $\pi_i = \pi'_i$, then $F(\pi) \wedge z_i = F(\pi') \wedge z_i$. This is summand compatibility. ∎

Lemma 2.4 If F is neutral monotone and not F^0 then F is summand compatible.

Proof: By [5], Proposition 2.3, p. 60, F is Paretian. Also, F is isotone and decisive. in view of Lemma 2.3, we need only show that F is a meet homomorphism.
Using F isotone, $F(\pi \wedge \pi') \leq F(\pi) \wedge F(\pi')$. Suppose there exist profiles π, π' for which $F(\pi \wedge \pi') < F(\pi) \wedge F(\pi')$. Since L is atomistic, there must exist an atom c such that $c \not\leq F(\pi \wedge \pi')$ and $c \leq F(\pi) \wedge F(\pi')$

Assuming $\pi = (x_1, x_2, \ldots, x_n)$ and $\pi' = (x'_1, x'_2, \ldots, x'_n)$ and the fact that F is Paretian, there must exist an index j such that $c \not\leq x_j \wedge x'_j$. Note that $N_c(\pi \wedge \pi') = N_c(\pi) \cap N_c(\pi') \subset$ both $N_c(\pi)$ and $N_c(\pi')$. Why? If, for example, $N_c(\pi \wedge \pi') = N_c(\pi)$, then by F neutral monotone, $c \leq F(\pi \wedge \pi')$, a contradiction.

Let z_i be the unique atom of $Z(L)$ for which $c \leq z_i$. Since $[0, z_i]$ is simple with cardinality greater than 2, there must exist an atom $b \leq z_i$ such that $b\delta c$. In other words, such that for some $x \in [0, z_i]$, $b, c \not\leq x$ and $b < c \vee x$. Observe that $c \not\leq x$ by definition of δ, and $x \not\leq c$.

Following a proof given in [4], we now define a profile π'':

$x''_j = c$ for $c \in \pi(j)$ but $\notin \pi(j) \wedge \pi'(j)$.

$x''_j = x$ for $c \in \pi'(j))$ but $\notin \pi(j) \wedge \pi'(j)$.

$x''_j = c \vee x$ for $c \in \pi(j) \wedge \pi'(j)$. (Not needed if $\pi(j) \wedge \pi'(j) = 0$)

$x''_j = 0$ otherwise

1. Note that $j \in N_c(\pi'') \iff c \leq x''_j \iff j \in N_c(\pi)$. Thus $N_c(\pi'') = N_c(\pi)$.
2. A similar argument shows that $N_x(\pi'') = N_c(\pi')$.
3. Also, $j \in N_b(\pi'') \iff b \leq x''_j \iff x''_j = c \vee x \iff c \in \pi \wedge \pi'$. Thus $N_b(\pi'') = N_c(\pi \wedge \pi')$.
4. $N_c(\pi'') = N_c(\pi)$ and $c \leq F(\pi)$ imply by F neutral that $c \leq F(\pi'')$.
5. $N_x(\pi'') = N_c(\pi')$ with $c \leq F(\pi')$, so $x \leq F(\pi'')$ by Lemma 4 of [4].
6. Then $b < c \vee x \leq F(\pi'')$ and $N_c(\pi \wedge \pi') = N_b(\pi'')$ implies that $c \leq F(\pi \wedge \pi')$ (using neutrality), a contradiction. If $N_c(\pi \wedge \pi') = \emptyset$, the contradiction is reached using vacuous implication. ∎

We now may restate Theorem 2.5, questioning whether summand compatibility can be removed from (P1). We leave the details to the reader.

Theorem 2.5 Each of the following conditions are equivalent:

(P1') F is summand compatible, decisive and Paretian.

(P2') F is neutral monotone but is not F^0.

(P3') F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \pi(j)$ for any profile π.

(P4') F is a residual map and $F(\pi_a) \geq a$ for every atom a.

(P5') F is a generalized oligarchy.

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