Addendum to “Generalized Oligarchies”

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1 Example for an Oligarchy

Example 1.1 This is an artificial example designed to help the reader’s intuition for understanding the possible practical use of a lattice theoretic approach to oligarchies. The primary references are [1] and [2]. There was a shocking and well publicized pair of explosions on April 13, 2013 that occured in Boston, MA near the finish line of the Boston Marathon. The author has no direct knowledge of the events prior to or subsequent to the events, but will present instead a fictional event from which one can gain intuition.

The example: There is a massive explosion near the finish line of a race involving hundreds of runners. People are killed or injured. Surveilance cameras have photographs of possible suspects who may or may not have planted the bombs.

The police have a number of decisions they need to quickly make. Among them are:

(1) Degree of public involvement:
Broadcast the photos?
(a) Newspaper? (b) TV? (c) Posters? (d) No publicity (1) All possible? (0) No advice

A committee of five experts is appointed to advise the local police. A rather crude lattice approach can be devised as follows: Let $L = \{0, a, b, c, d, 1\}$ with a, b, c, d atoms and $0, 1$ the smallest and largest elements. More accurate lattice models might involve the partitions or the weak orders on a finite set.

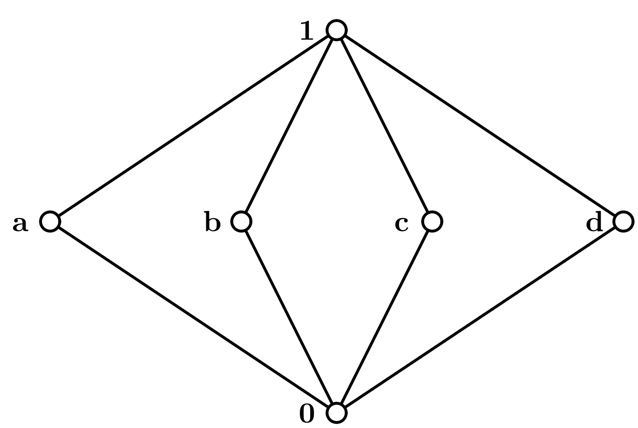

Figure 1: The lattice $L(a, b, c, d, 1)$ of choices.

In other situations we will specify the atoms and the largest element of an appropriate lattice constructed along the lines of Figure 1. Here is a word about the represented choices. The meaning of choices a, b, c, d should be obvious. Think of 0 as denoting a vote for not giving advice and 1 as a vote for doing all of $\{a, b, c\}$. For $F: L^5 \to L$ a residual map with $F(\pi_x) \geq x$ and $x \in \{a, b, c\}$, we will indicate just how an oligarchy is produced. Recall that residual maps preserve 1 and are meet homomorphisms. Associated with any residual map $F : L^n \to L$ there is an associated residuated map $G: L \to L^n$ given by $G(x) = \bigwedge\{\pi: F(\pi) \geq x\}$. Recall further that a residuated map is a join homomorphism that preserves 0. Thus for x any atom of L. $G(x) \subseteq \pi_x$, so $G(x) = (x_1, x_2, x_3, x_4, x_5)$, where $x_i = 0$ or x. What makes all this work is that the indices for which $x_i = x$ are independent of the value of x. Thus if $G(a) = (a, 0, a, a, 0)$ then $G(b) = (b, 0, b, b, 0)$, $G(c) = (c, 0, c, c, 0)$, $G(d) = (d, 0, d, d, 0)$ and in general $G(\pi) = \pi(1) \wedge \pi(3) \wedge \pi(4)$ for any profile π. Thus, for example, $F(a, 1, 1, b, 1) = 0$ and $F(a, 1, 1, 1, a) = a$.

Technical details appear in [**?**] and [2].

2 Extension to a Generalized Oligarchy

Example 2.1 Let us extend the fictional example.
Here are two possible areas of potential concern to the police. The first example is Example 1.1 using the issues raised in the last section but with the lattice of choices $L(a, b, c, d, z_1)$. Here is the next example:

(2) Should a reward be offered?

(e) Small? (f) Large (g) No reward (0) No advice (z_2) By all means a reward, but no advice on size.

The lattice we use here is $L(e, f, g, z_2)$. The labels are as expected, but the lattice we shall now use is L equals the internal direct sum of $[0, z_1]$ with $[0, z_2]$. The reader should recall the natural isomorphism between this lattice L and the external direct product $[0, z_1] \times [0, z_2]$ given by $x \vee y \leftrightarrow (x, y)$ for $x \leq z_1$ and $y \leq z_2$. We shall alternate between the two versions. Then $Z(L)$ is 2^2 with atoms z_1 and z_2. Let $F: L^5 \to L$ be a generalized consensus function, and note that F induces two consensus functions F_1 on $[0, z_1]$ and F_2 on $[0, z_2]$. Take F_1 as in Example 1.1, and F_2 so that $F_2(\pi_e) = (e, 0, 0, 0, e)$.

Now look at an illustrate profile from L^5. It is formed from $\pi_{(1)} = (a, 0, z_1, a, b)$ and $\pi_{(2)} = (z_2, f, f, f, g)$ and this leads to $\pi = (a \vee z_2, f, z_1 \vee f, b \vee f, b \vee g)$. The reader should verify that this produces $F(\pi) = F_1((\pi(1)) \cup F_2(\pi(2)) = \{a, g\}$. In general for any profile π of L^5, $F(\pi) = F_1(\pi_{z_1} \wedge \pi) \cup F_2(\pi_{z_2} \wedge \pi)$.

Of course we could always just view the problems that led to F_1 and F_2 as separate consensus problems without any need to form a generalized consensus. The reason for generalized consensus theory is that one hopes that this idea might extend to more general lattices. For example, it might be worth looking at a finite atomistic lattice in which the ∇ relation is not symmetric, and in which there are k distinct standard elements (see [**?**]. Theorem 3.2 for a definition) whose join is 1, and which are atoms in the distributive lattice formed by the standard elements of L.

References

[1] Chambers, C. P. and Miller, A. D., *Rules for aggregating information*, Social Choice and Welfare **36**, 2011, 75–82.

[2] Leclerc, B. and Monjardet, B., *Aggregation and Residuation*, Order **30**, 2013, 261–268.