Generalized Oligarchies

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Background

L is a finite lattice.

Partially ordered set.

All a, b have a join ($a \vee b$) and meet ($a \wedge b$).

L represents possible actions or decisions.

$$L^n = L \times L \times \cdots \times L \text{ (n factors)}$$

Defn: A profile $\pi = (x_1, x_2, \ldots, x_n)$.

Idea: You are getting advice from n experts.

Entry x_i is advice from expert i.

Defn: A consensus function is a mapping $F\colon L^n \to L$.

$F(\pi)$ yields the summary advice.

Can think of L as representing partitions of a set, or weak orders, or at least three atoms representing choices with added 0 and 1.

These are all atomistic simple finite lattices.

atom: An element that covers 0.

atomistic: Every element is join of atoms.

simple: Only trivial congruences.

Conditions to Consider

$F\colon L^n \to L$ where L is a finite atomistic lattice.
$J =$ set of atoms.
Terminology:
For profile π and $a \in L$, define $N_a(\pi) = \{i\colon a \leq \pi(i)\}$.
For $x \in L$, $\pi_x = (x, x, \ldots, x)$.
Define F^0 by $F^0(\pi) = 0$ for all profiles π.
Paretian: $N_a(\pi) = N \Rightarrow a \leq F(\pi)$.
Decisive: If $N_a(\pi) = N_a(\pi')$, then $a \leq F(\pi) \Leftrightarrow a \leq F(\pi')$.
Neutral monotone: For $a, a' \in L$, if $N_a(\pi) \subseteq N_{a'}(\pi')$,
then $a \leq F(\pi) \Rightarrow a' \leq F(\pi')$.
Oligarchy: $\exists M \subseteq N$ such that $F(\pi) = \bigwedge\{\pi(j)\colon j \in M\}$.
Think of appointing a committee M that jointly acts as a dictator.
Residual map $F(\pi_1) = 1$ and F is a meet homomorphism.

Fundamental Theorem

Theorem: (Leclerc and Monjardet) L is a finite simple atomistic lattice with cardinality > 2. $F\colon L^n \to L$. Following are equivalent:

1. F is decisive and Paretian.
2. F is neutral monotone but not F^0.
3. F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \{\pi(j)\}\ \forall \pi$.
4. F is a residual map and $F(\pi_a) \geq a$ for all atoms a.
5. F is an oligarchy.

From *Aggregation and Residuation*, Order **30**, 2013, 261–268.
Wish to extend this to direct products of simple lattices.
Inspiration: Boston Marathon bombing, or weather events like Hurricane Sandy or the World Trade Center attack.

Finite Atomistic Lattice L

Motivation: Want to apply Oligarchies to several problems at the same time. They may or may not independently reach their decisions.

External idea: Take $L_1, L_2, \ldots, L_k$ to be finite nondistributive atomistic simple lattices, with $L = L_1 \times L_2 \times \cdots \times L_k$.
Define consensus functions F_i on L_i each with the same value of n.
Let π_i be profile on $(L_i)^n$ for each i with $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$.
Define $F\colon L^n \to L$ by $F(\pi) = (F_1(\pi_1), F_2(\pi_2), \ldots, F_k(\pi_k))$.

More Terminology:
Defn: For $a, b \in L$, write $a \nabla b$ if $(a \vee x) \wedge b = x \wedge b \; \forall x$.
Defn: For a, b atoms write $a \delta b$ if $a \neq b$ and for some $x \in L$, $a < b \vee x$ and $a, b \not\leq x$.
∇ and δ:
Fact: For a, b distinct atoms, $a \nabla b$ fails $\Leftrightarrow$ $b \delta a$.
Proof: $(a \vee x) \wedge b > x \wedge b$ means $b \leq a \vee x$ and $b \not\leq x$.

Let δ^t denote transitive closure of δ.

Fact: L is simple iff every pair of atoms is connected by δ^t.

Defn: s in a lattice is standard if $\forall x, y$, $(s \vee x) \wedge y = (s \wedge y) \vee (x \wedge y)$. s induces a congruence Θ_s by $x \Theta_s y$ if $x \vee y = (x \wedge y) \vee s_1$ for some $s_1 \leq s$.

Fact: Every congruence on a finite atomistic lattice is generated by a standard element.

Fact: If ∇ is symmetric, then every congruence is generated by a central element z.

Defn: z is central iff it has a complement z' and L is isomorphic to $[0, z] \times [0, z']$ under the mapping $x \mapsto (x \wedge z, x \wedge z')$.

Theorem: If ∇ is symmetric, then L is the direct product of simple atomistic lattices.

Generalized oligarchies

$F\colon L^n \to L$ where L is a finite atomistic lattice that is a direct product of k simple lattice each with cardinalty ≥ 3.

Let $z_1, z_2, \ldots, z_k$ be the atoms of the center of L, so each $[0, z_i]$ is simple.

Defn:For each profile π, let $\pi_i = \pi \wedge \pi_{z_i}$.

Defn:For each z_i, define F_i on $[0, z_i]$ by $F_i(\pi_i) = F(\pi) \wedge z_i$.

If $\pi_i = \pi'_i \, \forall i$, then $\pi = \pi'$, and there is no issue. For this to make sense for a single index i, need F summand compatible in that $\pi_i = \pi'_i$ implies $F(\pi) \wedge z_i = F(\pi') \wedge z_i$.

Lemma: If $F(\pi \wedge \pi_{z_i}) = F(\pi) \wedge F(\pi_{z_i})$ and $F(\pi_{z_i}) \geq z_i$,
or if F is neutral monotone and not F^0,
or if $F(\pi_{z_i}) = z_i \, \forall i$,
then F is summand compatible.

Distributive simple atomistic lattices have cardinality ≤ 2, so L is the direct product of a Boolean lattice and some simple lattices each having cardinality > 2.

Theorem: (Improved result) L is a finite atomistic lattice that is the direct product of simple lattices each having cardinality > 2. $F\colon L^n \to L$. Following are equivalent:

1. F is decisive, Paretian and summand compatible.
2. F is neutral monotone but not F^0.
3. F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \{\pi(j)\}\ \forall \pi$.
4. F is a residual map and $F(\pi_a) \geq a$ for all atoms a.
5. F is a *generalized oligarchy* in the sense that for every atom z_i of the center of L, each induced consensus function F_i defined on $[0, z_i]$ by $F_i(\pi \wedge \pi_{z_i}) = F(\pi) \wedge z_i$ is an oligarchy.

Detour: Residuated and Residual

Let P, Q be finite lattices.
residual $F\colon P \to Q$: meet homomorphism $F(1) = 1$.
residuated $G\colon Q \to P$: join homomorphism and $G(0) = 0$.

For F residual, there is an associated residuated G defined by

$$G(q) = \bigwedge\{p \in P\colon q \leq F(p).\} \ .$$

Linked by: $p \leq FG(p)$ and $q \geq GF(q) \, \forall p \in P, \ q \in Q$.
The setting: L is a finite simple atomistic lattice having cardinality at least 3, and $F\colon L^n \to L$ is a residual map such that for every atom a of L, $F(\pi_a) \geq a$.
Proof of how F gets to be an oligarchy.
$G\colon L \to L^n$ is the residuated map associated with F.
Apply G to $a \leq F(\pi_a)$ to obtain $G(a) \leq GF(\pi_a) \leq \pi_a$.
Thus for any index j, $G_j(a) \in \{0, a\}$.

Here G_j is the jth component of G.

Defn: Let $M(a) = \{j \in N\colon G_j(a) = a\}$.

Residual maps and Oligarchies

Fact: For distinct atoms a and b, $a\delta b \Rightarrow M(a) \subseteq M(b)$.

Proof: $a\delta b$ implies $\exists x \in L$ such that $a < b \vee x$ and $a \not\leq x$.
Using L atomistic, $\exists$ finite family of atoms K such that $a \leq \bigvee K$, $a \notin K$, while $b \in K$. We may clearly assume K is such a family having minimal cardinality. Then $a \leq \bigvee K$.
Applying the residuated mapping G to this inequality, and $G_j = jth$ component of G, $G_j(a) = a \Rightarrow G_j(c) = c \, \forall c \in K$. ∎

Fund. Fact: If L is simple, then $M(a) = M(b)$ for all atoms a, b.
Now if $M = M(a)$ for any atom a, then
$a \leq F(\pi) \Leftrightarrow G(a) \leq GF(\pi) \leq \pi$. Hence for each index $j \in M$,
$a = G_j(a) \leq \pi(j)$, so $a \leq \pi(j)$ for all $j \in M$, and
$a \leq F(\pi) \Leftrightarrow a \leq \pi(j) \, \forall j \in M \Leftrightarrow a \leq \bigwedge\{\pi(j) \colon j \in M\}$.
Since L is atomistic, it follows that $F(\pi) = \bigwedge\{\pi(j) \colon j \in M\}$.

So we see how residuated maps are the key to constructing an oligarchy on a finite simple atomistic lattice.
(Proof due to Monjardet and Leclerc)

Future Projects

- Suppose there is a meet homomorphism from a finite atomistic lattice L onto a direct product of simple lattices each having cardinality > 2. What then?
- Does any of this extend to finite subdirect products of simple lattices, each having cardinality > 2.
- What about finite lattices that are not atomistic or atomistic but not ∇-symmetric?
- Does a lattice theoretic approach yield any insight into other consensus functions?

That's all Folks!