Generalized Oligarchies and Congruences on finite lattices

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Internet Version:
http://home.dimacs.rutgers.edu/∼melj/Colloq3tr.pdf

Talk will be in two parts

1. First part will discuss a connection between some social choice conditions and some conditions that involve the structure of finite lattices. Intuition comes from partitions of a finite set or the subsets of a finite set ordered by set inclusion.
2. The second part involves a set-theoretic view of the structure of the congruences on a finite lattice. (Time permitting)

Lattice Theory Background

L is a finite lattice.

Partially ordered set.

All a, b have a join ($a \vee b$) and meet ($a \wedge b$).

L is a finite atomistic lattice (unless otherwise specified).

atom: An element that covers 0.

atomistic: Every element is join of atoms.

simple lattice: Only has trivial congruences.

congruence Θ : Equivalence relation such that

$x \Theta y \Rightarrow x \vee t \,\Theta\, y \vee t$ and $x \wedge t \,\Theta\, y \wedge t \ \forall t \in L$.

Trivial congruences $x \Theta_1 y$ if $x = y$ or $x \Theta_2 y \ \forall x, y$.

Interval: If $a \leq b$, then $[a, b] = \{x \in L \colon a \leq x \leq b\}$.

Why, Oh Why?

Why did I ever look at this material?
Paper by Chambers and Miller on Oligarchies.
Appeared in Social Choice and Welfare **36**, 2011.
Tied together social welfare conditions with pure lattice theory and referred to a book I helped write in 1972.
Results applied to lattice of partitions.
Extended to finite simple atomistic lattices by Leclerc and Monjardet in ORDER, 2013.

Got me looking again at old papers I helped write. Led me to look at original papers that involved congruences.

Finite Atomistic Lattice L

For $a, b \in L$, write $a \nabla b$ if $(a \vee x) \wedge b = x \wedge b \; \forall x \in L$.

∇ was originally studied in conjnction with internal direct product decomposition of atomistic lattices.

For a, b atoms write $a \delta b$ if $a \neq b$ and for some $x \in L$, $a < b \vee x$ and $a, b \nleq x$.

∇ and δ: For a, b distinct atoms, $a \nabla b$ fails $\Leftrightarrow b \delta a$.

Proof: $(a \vee x) \wedge b > x \wedge b$ means $b \leq a \vee x$ and $a, b \nleq x$. ∎

Fact: For atoms a, b with $a \delta b$: $b \Theta 0$, $a \Theta b \Rightarrow a \Theta 0$.

Defn: δ^t is transitive closure of δ.

Defn: L simple means there are at most two congruences on L.

Fact: L is simple iff every pair of atoms is connected by δ^t.

Congruences

Defn: $s \in L$ is standard if $(s \vee x) \wedge y = (s \wedge y) \vee (x \wedge y) \, \forall x, y$.
For a finite atomistic lattice every congruence is generated by a standard element s.
$x \Theta_s y \Leftrightarrow x \vee y = (x \wedge y) \vee s_1$ for some $s_1 \leq s$.

Fact: If ∇ is symmetric, then every congruence is generated by a central element z. So the congruences form a Boolean lattice.

Defn: z is central iff it has a complement z' and L is isomorphic to $[0, z] \times [0, z']$ under $x \mapsto (x \wedge z, x \wedge z')$.
Thus $x \mapsto x \wedge z$ is a homomorphism of L onto $[0, z]$.

Direct products

Fact: For L atomistic, $x \nabla y \Leftrightarrow p \nabla q$ for all atoms $p \leq x$, $q \leq y$. Thus ∇ is completely determined by its behavior on pairs of atoms.

Defn: L is dual atomistic iff every element is the meet of a family of dual atoms. A dual atom is covered by the largest element 1.

Theorem In any finite dual atomistic lattice, the following are equivalent:

- $a \nabla b$
- $x = (x \vee a) \wedge (x \vee b)\ \forall x \in L$.
- $a \vee x = 1 \Rightarrow b \leq x$.

So $a \nabla b$ implies $b \nabla a$.

There is both an internal and external version of
direct product decompositions of L.

Internal: Look at a central z with complement z'.

External: Look at ordered pairs (x, y) with $x \leq z$, $y \leq z'$.

The way it once was

Defn: a is *perspective* to b ($a \sim b$). The transitive closure is called a *projective* to b and denoted $a \approx b$. Then $a \sim b$ means $a \vee x = b \vee x$ with $a \wedge x = b \wedge x = 0$ for some $x \in L$.

Fact: If L is atomistic and dual atomistic, then for the atoms a, b:

$a \nabla b$ fails iff $a \sim b$.

Hence L is simple iff $a \approx b$ for all atoms a, b.

Assume L is a finite atomistic lattice in which $a \nabla b \Rightarrow b \nabla a$.

Theorem: L is a direct product of simple lattices.

A simple lattice is distributive iff it has cardinality ≤ 2. Group the distributive simple factors. They form a Boolean lattice.

Theorem: L is either a Boolean lattice or it is simple with $a \delta^t b$ for all pairs a, b of atoms, or it is a direct product of such lattices.

Background for Oligarchies

L is a finite lattice. $N = \{1, 2, \ldots, n\}$.
L represents possible actions or decisions.

$$L^n = L \times L \times \cdots \times L \ (n \text{ factors})$$

Defn: A profile $\pi = (x_1, x_2, \ldots, x_n)$.
Idea: You are getting advice from n experts.
Entry x_i is advice from expert i.
Defn: A consensus function is a mapping $F\colon L^n \to L$.
$F(\pi)$ yields the summary advice.
Can think of L as representing partitions of a set, or weak orders, or atoms representing choices with added 0 and 1. These are all simple finite lattices that are both atomistic and dual atomistic.

Conditions to Consider

$F\colon L^n \to L$ where L is a finite atomistic lattice.
J = set of atoms.
Terminology:
For profile π and $a \in L$, define $N_a(\pi) = \{j\colon a \leq \pi(j)\}$.
For $x \in L$, $\pi_x = (x, x, \ldots, x)$.
Define F^0 by $F^0(\pi) = 0$ for all profiles π.
Paretian $N_a(\pi) = N \Rightarrow a \leq F(\pi)$.
Decisive If $N_a(\pi) = N_a(\pi')$, then $a \leq F(\pi) \Leftrightarrow a \leq F(\pi')$.
Neutral monotone For $a, a' \in L$, if $N_a(\pi) \subseteq N_{a'}(\pi')$,
then $a \leq F(\pi) \Rightarrow a' \leq F(\pi')$.
Oligarchy $\exists M \subseteq N$ such that $F(\pi) = \bigwedge\{\pi(j)\colon j \in M\}$.
Residual map $F(\pi_1) = 1$ and F is a meet homomorphism.

Fundamental Theorem

Theorem (Leclerc and Monjardet) L is a finite simple atomistic lattice with cardinality > 2. $F\colon L^n \to L$. Following are equivalent:

1. F is decisive and Paretian.
2. F is neutral monotone but not F^0.
3. F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \{\pi(j)\}\ \forall \pi$.
4. F is a residual map and $F(\pi_a) \geq a$ for all atoms a.
5. F is an oligarchy.

From *Aggregation and Residuation*, Order **30**, 2013, 261–268.
Wish to extend this to direct products of simple lattices.
Inspiration Boston Marathon bombing, or a weather event like Hurricane Sandy or the World Trade Center attack.

The Idea

Take $L_1, L_2, \ldots, L_k$ to be finite atomistic simple lattices each with cardinality > 2, with $L = L_1 \times L_2 \times \cdots \times L_k$.

Define consensus methods F_i on L_i each with the same value of n. For each i, let π_i be a profile on L_i.

Define π on L by $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$, and $F(\pi) = (F_1(\pi_1), F_2(\pi_2), \ldots, F_k(\pi_k))$.

This is an external version of what is planned, We still need an internal version.

Generalized oligarchies

$F\colon L^n \to L$ where L is a finite atomistic lattice that is a direct product of k simple lattices, each with cardinalty ≥ 3.
Let $z_1, z_2, \ldots, z_k$ be atoms of the center of L, so each $[0, z_i]$ is simple.
For each profile π, let $\pi_i = \pi \wedge \pi_{z_i}$.

For each z_i, define F_i on $[0, z_i]$ by $F_i(\pi_i) = F(\pi) \wedge z_i$.
If $\pi_i = {\pi'}_i \, \forall i$, then $\pi = \pi'$ and there is no problem.

For this to make sense for a given i, need F summand compatible in that $\pi_i = \pi'_i$ implies $F(\pi) \wedge z_i = F(\pi') \wedge z_i$.

Lemma: If $F(\pi \wedge \pi_{z_i}) = F(\pi) \wedge F(\pi_{z_i})$ and $F(\pi_{z_i}) \geq z_i$,
or if F is neutral monotone and not F^0,
or if $F(\pi_{z_i}) = z_i \, \forall i$
then F is summand compatible.

Improved Result

Theorem L is a finite atomistic lattice that is the direct product of simple lattices each having cardinality > 2. $F\colon L^n \to L$. Following are equivalent:

1. F is decisive, Paretian and summand compatible.
2. F is neutral monotone but not F^0.
3. F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \{\pi(j)\}\ \forall \pi$.
4. F is a residual map and $F(\pi_a) \geq a$ for all atoms a.
5. F is a generalized oligarchy in the sense that for every atom z_i of the center of L, each induced consensus funnction F_i defined on $[0, z_i]$ by $F_i(\pi \wedge \pi_{z_i}) = F(\pi) \wedge z_i$ is an oligarchy.

Can this result be further improved?

Detour: Residuated and Residual

Let P, Q be finite lattices.
residual: $F\colon P \to Q$: meet homomorphism $F(1) = 1$.
residuated: $G\colon Q \to P$: join homomorphism and $G(0) = 0$.

For F residual, there is an associated residuated G defined by

$$G(q) = \bigwedge\{p \in P\colon q \leq F(p).\}$$

Linked by $p \leq FG(p)$ and $q \geq GF(q) \forall p \in P,\ q \in Q$.
The setting: L is a finite simple atomistic lattice having cardinality at least 3, and $F\colon L^n \to L$ is a residual map such that for every atom a of L, $F(\pi_a) \geq a$.
proof: How F gets to be an oligarchy.
$G\colon L \to L^n$ is the residuated map associated with F.
Apply G to $a \leq F(\pi_a)$ to obtain $G(a) \leq GF(\pi_a) \leq \pi_a$.
Here $G(a) = (G_1(a), G_2(a), \ldots, G_k(a))$ where each $G_i(a) \in \{0, a\}$.
Defn: Let $M(a) = \{i \in N\colon G_i(a) = a\}$.

Residual maps and Oligarchies

Theorem: For distinct atoms a and b, $a\delta b \Rightarrow M(a) \subseteq M(b)$.
Proof: $a\delta b$ implies $\exists x \in L$ such that $a < b \vee x$ and $a \not\leq x$.
Using L atomistic, $\exists$ finite family of atoms K such that $a \leq \bigvee K$, $a \notin K$, while $b \in K$. We may clearly assume K is such a family having minimal cardinality. Then $a \leq \bigvee K$.
Applying the residuated mapping G to this inequality, with $G(a) = (G_1(a), G_2(a), \ldots, G_k(a))$, and $G_i =$ ith component of G, $G_i(a) = a \Rightarrow G_i(c) = c \, \forall c \in K$. ∎

Fund. Fact: If L is simple, then $M(a) = M(b)$ for all atoms a, b.
Now if $M = M(a)$ for any atom a,

then $a \leq F(\pi) \Leftrightarrow G(a) \leq GF(\pi) \leq \pi$. Hence for each coordinate $i \in M$, $a = G_i(a) \leq \pi(i)$, so $a \leq \pi(i)$ for all $i \in M$.

$a \leq F(\pi) \Leftrightarrow a \leq \pi(i) \, \forall i \in M \Leftrightarrow a \leq \bigwedge\{\pi(i) \colon i \in M\}$. Since L is atomistic, it follows that $F(\pi) = \bigwedge\{\pi(i) \colon i \in M\}$.

Future Projects for Oligarchies

Assume L a finite atomistic lattice.

- Suppose there is a meet homomorphism from L onto a finite direct product of simple lattices. What happens then?
- When does a generalized oligarchy become an oligarchy?
- Does any of this extend to subdirect products of simple lattices?
- What about finite lattices that are not atomistic?
- Does a lattice theoretic approach yield any insight into other consensus functions?

Second part of talk

General theory of congruences for finite lattice L

Join-irreducibles: Elements $j > 0$ such that
$j > j_* = \bigvee\{x \in L \colon x < j\}$. $J = J(L) =$ join-irreducibles.
Facts: Every element of L is join of the join-irreducibles below it.
Every Θ is determined by $\{j \in J \colon j\Theta j_*\}$.
Defn: For $p, q \in J$, qCp if $q < p \vee x, q \not\leq x \vee p_*$ for some x.
Thus qCp, $p\Theta p_* \Rightarrow q\Theta q_*$. (Idea from Alan Day)
Defn: Jset $\subseteq J$ is set K such that $p \in K, qCp \Rightarrow q \in K$.
$\widehat{C}$ is reflexive transitive closure of C, so $\widehat{C}$ is a quasiorder.
Defn: Write pEq if $p\widehat{C}q$ and $q\widehat{C}p$,
noting that $\widehat{C}/E$ is a partial order
Con(L) is isomorphic to the order ideals of this poset.
Details of proof not given.
Note: For L atomistic, $qCp \Leftrightarrow q\delta p$. (Monjardet)

Turns out this can be made abstract with easier proofs of more general results.

Set-theoretic Approach

J is a finite set (think of join-irreducibles of finite lattice).
$R_C \subseteq J \times J$ irreflexive (xRx fails $\forall x$)
    $qR_C p$ abstraction of qCp.
$R_{\widehat{C}}$ is reflexive transitive closure of R_C, and is a quasiorder.
    $qR_{\widehat{C}}p$ abstraction of $q\widehat{C}p$.
Defn: $\mathcal{V} = \{V \subseteq J \colon p \in V, qR_C p \Rightarrow q \in V\}$.
    $(\mathcal{V}, \subseteq)$ is a finite distributive lattice. $\{\emptyset, V\} \subseteq \mathcal{V}$.
    Elements of $\mathcal{V}$ called J-sets.
Fact: Smallest J-set containing $p \in \mathcal{V}$ is $V_p = \{q \in V \colon qR_{\widehat{C}}p\}$.
    These are the join-irreducibles of $\mathcal{V}$.
Fact: For $V \in \mathcal{V}$, $V = \bigcup\{V_p \colon p \in \mathcal{V}\}$.

Atoms: A is an atom of $\mathcal{V}$ then $A = V_p$ for all $p \in A$.
For any atom A, $p, q \in A \Rightarrow (p, q) \in R_{\widehat{C}} \cap R_{\widehat{C}}^{-1}$.

Complements: $P \in \mathcal{V}$ has complement in $\mathcal{V}$ iff $J \setminus P \in \mathcal{V}$.

Fact: If $R_{\widehat{C}}$ symmetric, then $\mathcal{V}$ is atomistic.

Fact: $R_{\widehat{C}}$ is symmetric if and only if $\mathcal{V}$ is a Boolean lattice.
Can $R_{\widehat{C}}$ be symmetric with R_C not symmetric?

Fact: For $P \in \mathcal{V},\ P^* = J \setminus R_{\widehat{C}}(P)$ (pseudocomplement of P.)

Defn: $\mathcal{V}$ is Stone lattice if every pseudocomplement has a complement.

Fact: $\mathcal{V}$ Stone lattice iff $R_{\widehat{C}}$ is such that for each $a \in V$, there is unique atom V_k of $\mathcal{V}$ such that $V_k \subseteq V_a$.

This also is when $\mathcal{V}$ is a direct product of subdirectly irreducible factors.

Thank you for listening to the very end!

That's all folks!