Chapter 1

Hypergeometric Series

1.1 The general hypergeometric series

Definition 1.1. A series \( \sum_{k=0}^{\infty} t_k \) is called hypergeometric if \( t_{k+1}/t_k \) is a rational function of \( k \).

Definition 1.2. The rising factorial \( \prod_{j=0}^{k-1} (a+j) = a(a+1)(a+2) \cdots (a+k-1) \) is denoted by the Pochhammer symbol \( (a)_k \).

Note that \( (a)_0 = 1 \) and \( k! = (1)_k \).

As we said hypergeometric series \( \sum_{k=0}^{\infty} t_k \) has the property that \( t_{k+1}/t_k \) is a rational function \( R(k) \). Without loss of generality, we may write \( R(k) \) in the form

\[
R(k) = \frac{(k+a_1)(k+a_2) \cdots (k+a_p)}{(k+1)(k+b_1) \cdots (k+b_q)} z,
\]

and use this form to motivate the following standard hypergeometric notation:

Definition 1.3.

\[
pFq \left[ a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q : z \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(k+b_1)_k \cdots (b_q)_k} z^k.
\]

We will assume that none of the \( b_j \) is a nonpositive integer, as this would cause there to be zeros in the denominator.

Example 1.4. 1.

\[
e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = _0F_0 \left[ - \mid - : z \right].
\]

2.

\[
\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = _0F_1 \left[ \frac{1}{2} \mid -\frac{z^2}{4} \right].
\]

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3. \[ \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z \, {}_{0}F_{1} \left[ \frac{-}{\frac{3}{2} \ ; \ -\frac{z^2}{4}} \right]. \]

4. \[ \sin^{-1} z = z \, {}_{2}F_{1} \left[ \frac{1}{2}, \frac{1}{2} ; \ z^2 \right], \quad |z| < 1 \]

5. \[ \frac{1}{1-z} = {}_{1}F_{0} \left[ \frac{1}{-} ; z \right], \quad |z| < 1 \]

Notice that if at least one of the \(a_j\) is a negative integer, say \(-n\), then for all \(k > n\), the term \(t_k = 0\), and thus the series has only finitely many nonzero terms, so the question of convergence of the series does not arise.

On the other hand, if none of the \(a_j\) is a negative integer and \(z \neq 0\), then the series has infinitely many nonzero terms.

**Theorem 1.5.** The series \( \, {}_{p}F_{q} \left[ \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} ; z \right] \)

- converges absolutely for all \(z\) if \(p < q + 1\),
- diverges for all \(z \neq 0\) if \(p > q + 1\),
- converges absolutely for \(|z| < 1\) if \(p = q + 1\),
- diverges for \(|z| > 1\) if \(p = q + 1\).

**Proof.** Apply the ratio test from elementary calculus. \(\square\)

The ratio test, however, provides no convergence information when \(|z| = 1\) and \(p = q + 1\).

Let us now turn our attention to the particular case where \(p = q + 1 = 2\).

### 1.2 Gauß’s Hypergeometric Series

The hypergeometric series

\[ {}_{2}F_{1} \left[ \begin{array}{c} a, b \\ c \end{array} ; z \right] \quad (1.1) \]

was studied extensively by K. F. Gauß, and he delivered a famous lecture on such series in January 1812. By the results of the previous section, we know that (1.1) converges absolutely when \(|z| < 1\) and diverges for \(|z| > 1\). In order to study the case where \(|z| = 1\), we will need some background results.
1.2.1 Background material

**Definition 1.6 (Big “O” notation).** Let \( \zeta_1, \zeta_2, \zeta_3, \ldots \) and \( z_1, z_2, \ldots \) be a pair of sequences where \( \zeta_k / z_k < C \) for all sufficiently large \( k \) and \( C \) is a constant (independent of \( k \)). We then say that the sequence \( \{ \zeta_k \} \) is of order \( z_k \) and write \( \zeta_k = O(z_k) \).

**Example 1.7.**

\[ \frac{2^k}{k^3} = O \left( \frac{1}{k^2} \right) \]

because \( \frac{2^k}{k^3} \) is less than \( \frac{2}{k} \) for all \( k > 2 \).

**Proposition 1.8 (The Binomial Series).** For \( p \) constant and \( |z| < 1 \),

\[ (1 + z)^p = 1 + pz + O(z^2) \]

**Proof.** The Maclaurin series expansion of \( f(z) \) is

\[ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k, \]

so with \( f(z) = (1 + z)^p \), we have \( f(0) = 1, f'(0) = p \). The convergence follows by the ratio test. (Note that if \( p \) is a nonnegative integer, then the Maclaurin series reduces to a finite sum, and thus is valid for all \( z \).)

**Theorem 1.9 (De Morgan).** Let \( \sum_{k=1}^{\infty} u_k \) be a series whose terms satisfy

\[ \lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = 1. \]

The series \( \sum_{k=1}^{\infty} u_k \) converges absolutely if there exists a positive real number \( C \) such that

\[ \lim_{k \to \infty} k \left( \left| \frac{u_{k+1}}{u_k} \right| - 1 \right) = -(1 + C). \]

**Proof.** Let \( v_k = A k^{-(1+c/2)} \), where \( A \) is a constant. Clearly, \( \sum_{k=1}^{\infty} v_k \) is a convergent series (by the “p-series test” from elementary calculus). Notice that

\[ \frac{v_{k+1}}{v_k} = \left( \frac{k}{k+1} \right)^{1+c/2} = \left( 1 + \frac{1}{k} \right)^{-(1+c/2)} = 1 - \frac{1 + c/2}{k} + O \left( \frac{1}{k^2} \right), \]

where the last equality follows from Proposition 1.8 with \( z = 1/k \) and \( p = -(1 + c/2) \). Thus,

\[ \lim_{k \to \infty} k \left( \frac{v_{k+1}}{v_k} - 1 \right) = - \left( 1 + \frac{c}{2} \right). \]

A suitable choice of the constant \( A \) will ensure that for all \( k, \left| u_k \right| < v_k \). Since \( \sum v_k \) converges, so does \( \sum \left| u_k \right| \), and thus \( \sum u_k \) is absolutely convergent. \( \square \)
Corollary 1.10 (Raabe’s test). If

\[ \frac{|u_{k+1}|}{u_k} = 1 + \frac{B}{k} + O\left(\frac{1}{k^2}\right), \]

(where \(B\) is a constant independent of \(k\)), then \(\sum u_k\) is absolutely convergent if \(B < -1\).

1.2.2 The convergence of Gauß’s series when \(|z| = 1\).

Consider \(\sum_{k=0}^{\infty} t_k = 2F_1\left[\begin{array}{c} a, b \\ c \end{array} ; z\right]\) where \(|z| = 1\). The numbers \(a, b,\) and \(c\) are complex; let us write them as \(a = a_1 + i\alpha_2, \ b = b_1 + i\beta_2\) and \(c = c_1 + i\gamma_2\), where \(a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}\) and \(i = \sqrt{-1}\). Then we have

\[
\frac{|t_{k+1}|}{t_k} = \frac{|(k+a)(k+b)|}{(k+1)(k+c)} = \left|\frac{1 + \frac{a+b}{k}}{1 + \frac{1+c}{k}}\right|^2 \left|\frac{1+c}{k}\right| \left(1 + O\left(\frac{1}{k^2}\right)\right) = 1 + \frac{a_1+b_1-c_1-1 + i(a_2+b_2-c_2)}{k} + O\left(\frac{1}{k^2}\right).
\]

Thus by Corollary 1.10, the series converges absolutely if \(\Re(a+b-c) < 0\), where \(\Re z\) denotes the real part of the complex number \(z\). It turns out that even more is known than just whether or on the series converges in the \(z = 1\) case. Gauß gave an explicit formula for the sum of the series:

Theorem 1.11 (Gauß’s Hypergeometric Summation Formula). If \(\Re(a+b-c) < 0\),

\[ 2F_1\left[\begin{array}{c} a, b \\ c \end{array} ; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \]

Sketch of proof. First, show that

\[ 2F_1\left[\begin{array}{c} a, b \\ c \end{array} ; 1\right] = \frac{(c-a)(c-b)}{c(c-a-b)} 2F_1\left[\begin{array}{c} a, b \\ c+1 \end{array} ; 1\right]. \]

and more generally that

\[ 2F_1\left[\begin{array}{c} a, b \\ c \end{array} ; 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} 2F_1\left[\begin{array}{c} a, b \\ c+n \end{array} ; 1\right]. \]
for $n \in \mathbb{Z}_+$. Then show that
\[
\lim_{n \to \infty} \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}
\]
and
\[
\lim_{n \to \infty} {}_2F_1 \left[ \frac{a, b}{c+n} ; 1 \right] = 1.
\]

If $a = -n$ where $n$ is a positive integer, then Theorem 1.11 simplifies to

**Corollary 1.12 (Chu-Vandermonde).**
\[
{}_2F_1 \left[ \frac{-n,b}{c} ; 1 \right] = \frac{(c-b)_n}{(c)_n}.
\]

Of course, this result follows easily as a consequence of Theorem 1.11. We choose to present an independent proof, however.

**Proof.** Let
\[
F(n,k) := \begin{cases} 
\frac{(-n)_k(b)_n(c)_n}{k!(c)_k(c-b)_n} & \text{if } k \geq 0 \\
0 & \text{if } k < 0
\end{cases},
\]
l let
\[
R(n,k) := \frac{k(1-c-k)}{n(n+c-1)},
\]
let
\[
G(n,k) := F(n,k)R(n,k),
\]
and let
\[
f(n) := \sum_{k=0}^{\infty} F(n,k) = \frac{(c)_n}{(c-b)_n} {}_2F_1 \left[ \frac{-n,b}{c} ; 1 \right].
\]

\[
\frac{kc + kn - kb + nb - k}{n(n+c-1)} = \frac{kc + kn - kb + nb - k}{n(n+c-1)}
\]
\[
\iff \frac{(c+n-1)n - (n-k)(c+n-b-1)}{n(c+n-1)} = \frac{(b+k)(n-k) + k(k-1+c)}{n(n+c-1)}
\]
\[
\iff 1 - \frac{(n-k)(c+n-b-1)}{n(c+n-1)} = \frac{(n-k)(b+k)(k+1)(k+c)}{n(n+c-1)(k+1)(k+c)} + \frac{k(k-1+c)}{n(n+c-1)}
\]
\[
\iff 1 - \frac{F(n-1,k)}{F(n,k)} = \frac{F(n,k+1)R(n,k+1)}{F(n,k)} - R(n,k)
\]
\[
\iff F(n,k) - F(n-1,k) = F(n,k)R(n,k+1) - F(n,k)R(n,k)
\]
\[
\iff F(n,k) - F(n-1,k) = G(n,k+1) - G(n,k)
\]
\[
\implies \sum_{k=-\infty}^{\infty} \{F(n,k) - F(n-1,k)\} = \sum_{k=-\infty}^{\infty} \{G(n,k+1) - G(n,k)\}
\]
\[
\implies f(n) - f(n-1) = 0
\]
\[
\implies f(n) = f(n-1) \quad \text{for } n \in \mathbb{Z}_+.
\]
Thus \( f(n) \) is constant for all \( n \in \mathbb{Z}_+ \), so all we need to do to find \( f(n) \) for general \( n \) is to evaluate it at some particular value of \( n \), say \( n = 0 \).

\[
f(n) = f(0) = \sum_{k=0}^{\infty} \frac{(0)_k (b)_k (c)_{-k}}{k! (c - b)_k} = 1.
\]

Thus,

\[
\frac{(c)_n}{(c - b)_n} \binom{\sum_{k=0}^{\infty} \frac{(0)_k (b)_k (c)_{-k}}{k! (c - b)_k} - n, b}{c} 1 = 1,
\]

or, equivalently,

\[
\binom{\sum_{k=0}^{\infty} \frac{(0)_k (b)_k (c)_{-k}}{k! (c - b)_k} - n, b}{c} 1 = \frac{(c - b)_k}{(c)_k}.
\]

**Remark 1.13.** The novelty of the preceding proof is that it was produced automatically by a computer programmed to carry out the so-called “WZ algorithm” due to Wilf and Zeilberger. Zeilberger is professor here at Rutgers.

**Remark 1.14.** Corollary 1.12 was discovered in 1770 by the French mathematician Vandermonde and was called “Vandermonde’s sum” in the literature for many years. Recently, however, it was noticed that this identity had been discovered more than four and a half centuries earlier and had appeared in a book written in 1303 by the Chinese mathematician Chu Shih-Chieh, so we now call it the “Chu-Vandermonde sum.” This same book from 1303 contains an illustration of what we call “Pascal’s triangle” and refers to it as an “ancient method.”

**1.3 Exercises**

1. Prove that
\[
\frac{10 + 20k}{k^3} = O \left( \frac{1}{k^2} \right).
\]

2. Consider the series
\[
\sum_{k=0}^{\infty} \frac{k!}{(\alpha)_k}
\]
where \( \alpha \) is a positive real constant. Find the values of \( \alpha \) for which the series converges absolutely.

3. Find the Maclaurin series for \( \tan^{-1}(z) \), write the series in hypergeometric \( \binom{p}{q} \) notation, and find the set of \( z \) for which the series converges absolutely.
4. Find conditions on the complex numbers $a, b,$ and $c$ which cause

$$2F_1 \left[\begin{array}{c} a, b \\ c \\ i \end{array}\right]$$

to be absolutely convergent.

5. Show that

$$\frac{d^2}{dz^2} \left(2F_1 \left[\begin{array}{c} a, b \\ c \\ z \end{array}\right]\right) = \frac{a(a+1)b(b+1)}{c(c+1)} 2F_1 \left[\begin{array}{c} a+2, b+2 \\ c+2 \\ z \end{array}\right].$$

6. Prove that $y = 2F_1 \left[\begin{array}{c} a, b \\ c \\ z \end{array}\right]$ is a solution to the differential equation

$$z(1-z)\frac{d^2y}{dz^2} + \left[c - (a + b + 1)z\right]\frac{dy}{dz} - aby = 0.$$