§1.1, Exercise 11 Solution. *A local health food store packages three types of snack foods—Chewey, Cruncy, and Nutty–by mixing sunflower seeds, raisins, and peanuts.* First, we must assign the variables:

- Let $x_1 =$ the number of kg of SUNFLOWER SEEDS used in the CHEWEY mixture.
- Let $x_2 =$ the number of kg of SUNFLOWER SEEDS used in the CRUNCHY mixture.
- Let $x_3 =$ the number of kg of SUNFLOWER SEEDS used in the NUTTY mixture.
- Let $x_4 =$ the number of kg of RAISINS used in the CHEWEY mixture.
- Let $x_5 =$ the number of kg of RAISINS used in the CRUNCHY mixture.
- Let $x_6 =$ the number of kg of RAISINS used in the NUTTY mixture.
- Let $x_7 =$ the number of kg of PEANUTS used in the CHEWEY mixture.
- Let $x_8 =$ the number of kg of PEANUTS used in the CRUNCHY mixture.
- Let $x_9 =$ the number of kg of PEANUTS used in the NUTTY mixture.

Thus we immediately note the nonnegativity constraints $x_i \geqq 0$ for each $i = 1, 2, 3, \ldots, 9$.

In the textbook, the specifications for each mixture are given in a table, but here we will extract each piece of relevant information and translate it to a linear constraint immediately.

At least 60% of the CHEWEY must consist of RAISINS: This means that

$$\frac{x_4}{x_1 + x_4 + x_7} \geqq 0.6,$$

but this is not *linear*! However, we can can *convert* it into a linear inequality constrant by multiplying both sides of the inequality by denominator. Note that this is (sort of) legal since each of the nine variables is nonnegative and it is (probably) reasonable to assume that at least one of x_1, x_4, and x_7 will turn out to be nonzero. Thus the linear constraint we want is

$$x_4 \geqq 0.6(x_1 + x_4 + x_7),$$

which is equivalent to

$$0 \geqq 0.6x_1 - 0.4x_4 + 0.6x_7,$$

which is the same as

$$0.6x_1 - 0.4x_4 + 0.6x_7 \leqq 0.$$

We now proceed in an analogous fashion with the other four mixing requirements: *At most 20% of the CHEWEY must be PEANUTS:*

$$\frac{x_7}{x_1 + x_4 + x_7} \leqq 0.2 \Rightarrow x_7 \leqq 0.2(x_1 + x_4 + x_7) \Rightarrow 0.2x_1 + 0.2x_4 - 0.8x_7 \geqq 0.$$

At least 60% of the CRUNCHY must be SUNFLOWER SEEDS:

$$\frac{x_2}{x_2 + x_5 + x8} \geqq 0.6 \Rightarrow x_2 \geqq 0.6(x_2 + x_5 + x_8) \Rightarrow -0.4x_2 + 0.6x_5 + 0.6x_8 \leqq 0.$$

At most 20% of the NUTTY must be SUNFLOWER SEEDS:

$$\frac{x_3}{x_3 + x_6 + x_9} \leqq 0.2 \Rightarrow x_3 \leqq 0.2(x_3 + x_6 + x_9) \Rightarrow -0.8x_3 + 0.2x_6 + 0.2x_9 \geqq 0.$$

At least 60% of the NUTTY must be PEANUTS:

$$\frac{x_9}{x_3 + x_6 + x_9} \geqq 0.6 \Rightarrow x_9 \geqq 0.6(x_3 + x_6 + x_9) \Rightarrow 0.6x_3 + 0.6x_6 - 0.4x_9 \leqq 0.$$

This completes the constraints on the mixture formulations. Now, we turn our attention to the constraints on the amount of each type of mixture ingredient available every week:

The supplier of the ingredients can deliver each week...

- *at most 100 kg of sunflower seeds:*$\Rightarrow x_1 + x_2 + x_3 \leqq 100$
- *at most 80 kg of raisins:*$\Rightarrow x_4 + x_5 + x_6 \leqq 80$
- *at most 60 kg of peanuts:*$\Rightarrow x_7 + x_8 + x_9 \leqq 60$

Now let us turn our attention to the objective function: we want to maximize the profit, and the profit is the amount of money made from the sale of the mixtures *minus* the amount of money spent on the supplies, i.e. revenue minus cost. The problem tells us that *CHEWEY sells for \$2 per kg, CRUNCHY sells for \$1.60 per kg, and NUTTY sells for \$1.20 per kg*, thus the store's*revenue* (in dollars) is

$$2(x_1 + x_4 + x_7) + 1.6(x_2 + x_5 + x_8) + 1.2(x_3 + x_6 + x_9).$$

Next, the store's *cost*, we are told, is *\$1 per kg of sunflower seeds, \$1.50 per kg of raisins, and \$0.80 per kg of peanuts.* Thus the store's overall cost per week (in dollars) is

$$1(x_1 + x_2 + x_3) + 1.5(x_4 + x_5 + x_6) + 0.8(x_7 + x_8 + x_9).$$

We obtain the objective function (which measures profit) by taking revenue minus cost:

$$z = 2(x_1+x_4+x_7)+1.6(x_2+x_5+x_8)+1.2(x_3+x_6+x_9)-\Big(1(x_1+x_2+x_3)+1.5(x_4+x_5+x_6)+0.8(x_7+x_8+x_9)\Big).$$

Thus, the linear program which models the scenario described in §1.1, exercise 11 is:
max $z = x_1 + 0.6x_2 + 0.2x_3 + 0.5x_4 + 0.1x_5 - 0.3x_6 + 1.2x_7 + 0.8x_8 + 0.4x_9$
subject to

$$\begin{array}{rrrrrrrrrcr}
0.6x_1 & & & -0.4x_4 & & & +0.6x_7 & & & \leqq & 0 \\
0.2x_1 & & & +0.2x_4 & & & -0.8x_7 & & & \geqq & 0 \\
 & -0.4x_2 & & & +0.6x_5 & & & +0.6x_8 & & \leqq & 0 \\
 & & -0.8x_3 & & & +0.2x_6 & & & +0.2x_9 & \geqq & 0 \\
 & & 0.6x_3 & & & +0.6x_6 & & & -0.4x_9 & \leqq & 0 \\
x_1 & +x_2 & +x_3 & & & & & & & \leqq & 100 \\
 & & & x_4 & +x_5 & +x_6 & & & & \leqq & 80 \\
 & & & & & & x_7 & +x_8 & +x_9 & \leqq & 60 \\
x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8, & x_9 & \geqq & 0
\end{array}$$

Once you understand this problem, as further exercise make sure you know how to

1. rewrite the LP in standard and canonical forms, and
2. write the standard and canonical forms in matrix notation.