Polynomial Generalizations of q–Series Identities

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Srinivasa Ramanujan (1887–1920)

What is a q-analog?

The q–integers

- For positive integers *n*,

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$$

Heinrich Eduard Heine (1821–1881)

The *q*-integers

- For positive integers *n*,

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$$

The q-numbers

- For any complex number z,

$$[z]_q := \frac{1-q^z}{1-q}$$

Ordinary Derivative

$$Df(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

Ordinary Derivative

$$Df(x) := \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}$$

Ordinary Derivative

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Ordinary Derivative

$$Df(x) := \lim_{q \to 1} \frac{f(x) - f(qx)}{(1-q)x}$$

The *q*-Derivative

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}$$

Power rule for *q*-derivative

$$D_q x^n$$

Power rule for *q*-derivative

$$D_q x^n = \frac{x^n - (xq)^n}{x(1-q)}$$

Power rule for *q*-derivative

$$D_q x^n = \frac{1-q^n}{1-q} x^{n-1}$$

Power rule for *q*-derivative

$$D_q x^n = [n]_q x^{n-1}$$

The *q*–factorial

- For *n* a positive integer,

$$[n]_q! := [n]_q [n-1]_q [n-2]_q \ldots [2]_q [1]_q$$

The *q*-factorial

$$[0]_q! := 1$$

The exponential function

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

A q-exponential function

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}$$

A *q*–exponential function

$$D_q e_q^x = e_q^x$$

The binomial theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$$

The q–binomial

$$(x+y)_q^n := (x+y)(x+qy)(x+q^2y)\ldots(x+q^{n-1}y)$$

The q–binomial coefficient

- For $0 \leq j \leq n$,

$$\begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{[n]_q!}{[j]_q![n-j]_q!}$$

Karl Friederich Gauss (1777-1855)

The q–binomial theorem

$$(x+y)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} x^j y^{n-j}$$

Quantum binomial theorem

- If $yx = qxy$,

$$(x+y)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j y^{n-j}$$

The q–Pascal triangle

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_q$$

The *q*-Pascal triangle

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ j \end{bmatrix}_q$$

The *q*-binomial symmetry

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ n-j \end{bmatrix}_q$$

Finite fields

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- Let p be a prime and r be a positive integer.
- There exists a finite field (unique up to isomorphism) with $q=p^r$ elements.
- Denote this field GF(q).

The q–binomial coefficient

- The number of j–dimensional subspaces in the n–dimensional vector space $(\mathrm{GF}(q))^n$ is $\begin{bmatrix} n \\ j \end{bmatrix}_q$

Jacobi's triple product identity

- If $z \neq 0$ and $|w| < 1$, then

$$\sum_{j=-\infty}^{\infty} (-1)^j z^j w^{j^2} = \prod_{m=1}^{\infty} (1 - w^{2m})(1 - z w^{2m-1})(1 - z^{-1} w^{2m-1})$$

Karl Gustav Jacob Jacobi (1804–1851)

First Rogers–Ramanujan identity

- If $|q|<1$,

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

Leonard James Rogers (1862–1933)

First Rogers-Ramanujan identity

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

First Rogers-Ramanujan identity

$$1+q\left(\frac{1}{1-q}\right)+q^{4}\left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^{2}}\right)+\ldots$$

$$=\left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^{4}}\right)\left(\frac{1}{1-q^{6}}\right)\left(\frac{1}{1-q^{9}}\right)\ldots$$

First Rogers-Ramanujan identity

$$1 + q(1 + q + q^2 + \ldots) + q^4(1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots) + \ldots$$
$$= (1 + q + q^2 + \ldots)(1 + q^4 + q^8 + \ldots)(1 + q^6 + q^9 + \ldots)(1 + q^9 + q^{18} + \ldots)\ldots$$

First Rogers-Ramanujan identity

$$1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + \ldots$$

The Rogers-Ramanujan identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+2})(1-q^{5m+3})}$$

Rogers-Ramanujan type identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\ldots(1+q^{2j-1})}{(1-q^2)(1-q^4)\ldots(1-q^{2j})} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{8m+1})(1-q^{8m+4})(1-q^{8m+7})}$$

Rogers–Ramanujan type identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}(1+q+q^2)(1+q^2+q^4)\dots(1+q^j+q^{2j})}{(1-q)(1-q^2)\dots(1-q^{2j+1})} = \prod_{m=1}^{\infty} \frac{(1-q^{9m})}{(1-q^m)}$$

Freeman J. Dyson
(1923-   )

Rogers–Ramanujan type identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} = \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)(1-q)(1-q^3)\dots(1-q^{2j-1})}$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^{14m})(1-q^{14m-6})(1-q^{14m-8})}{(1-q^m)}$$

Let $\Sigma(q)$ denote the series side of the identity, in this case

$$\Sigma(q) = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)}$$

We seek a generalization $f(t,q)$ of $\Sigma(q)$

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- $f(t,q) = \sum_{n=0}^{\infty} P_n(q) t^n$ where $P_n(q)$ -->$\Sigma(q)$ and $P_n(q)$ are polynomials

We seek a generalization $f(t,q)$ of $\Sigma(q)$ where

- $$\lim_{t\to 1}(1-t)f(t,q)=\Sigma(q)$$
- $$f(t,q)=\sum_{n=0}^{\infty}P_n(q)t^n$$ where $P_n(q)$ -->$\Sigma(q)$ and $P_n(q)$ are polynomials
- $f(q,t)$ satisfies a first order nonhomogenous q–difference equation

$$f(t) := f(t,q) := \sum_{j=0}^{\infty} \frac{t^{2j} q^{j^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^{j-1})}$$

Deriving the *q*-difference eqn

$$f(t) = \sum_{j=0}^{\infty} \frac{t^{2j} q^{j^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^{j-1})}$$

Deriving the q-difference eqn

$$f(t) = \frac{1}{1-t} + \sum_{j=1}^{\infty} \frac{t^{2j} q^{j^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^{j-1})}$$

Deriving the q-difference eqn

$$f(t) = \frac{1}{1-t} + \sum_{j=0}^{\infty} \frac{t^{2(j+1)} q^{(j+1)^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^{j+1-1})}$$

Deriving the q–difference eqn

$$f(t) = \frac{1}{1-t} + \sum_{j=0}^{\infty} \frac{t^{2j+2} q^{j^2+2j+1}}{(1-t)(1-tq)(1-tq^2)\dots(1-tq^j)}$$

Deriving the *q*–difference eqn

$$f(t) = \frac{1}{1-t} + \frac{t^2 q}{1-t} \sum_{j=0}^{\infty} \frac{t^{2j} q^{j^2+2j}}{(1-tq)(1-tq^2)\ldots(1-tq^j)}$$

Deriving the *q*-difference eqn

$$f(t) = \frac{1}{1-t} + \frac{t^2 q}{1-t} \sum_{j=0}^{\infty} \frac{(tq)^{2j} q^{j^2}}{(1-tq)(1-tq^2)\ldots(1-tq^j)}$$

The *q*-difference equation

$$f(t) = \frac{1}{1-t} + \frac{t^2 q}{1-t} f(tq)$$

The *q*-difference equation

$$(1-t)f(t) = 1 + t^2 q f(tq)$$

The *q*–difference equation

$$f(t) - tf(t) = 1 + t^2 qf(tq)$$

The *q*-difference equation

$$f(t) = 1 + tf(t) + t^2 q f(tq)$$

The *q*–difference equation

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + t\sum_{n=0}^{\infty} P_n(q)t^n + t^2 q \sum_{n=0}^{\infty} P_n(q)(tq)^n$$

The *q*-difference equation

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=0}^{\infty} P_n(q)t^{n+1} + \sum_{n=0}^{\infty} P_n(q)t^{n+2}q^{n+1}$$

The *q*–difference equation

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=0}^{\infty} P_n(q)t^{n+1} + \sum_{n=0}^{\infty} q^{n+1} P_n(q)t^{n+2}$$

The recurrence for $P_n(q)$

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=1}^{\infty} P_{n-1}(q)t^n + \sum_{n=2}^{\infty} q^{n-1}P_{n-2}(q)t^n$$

The recurrence for $P_n(q)$

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=1}^{\infty} P_{n-1}(q)t^n + \sum_{n=2}^{\infty} q^{n-1}P_{n-2}(q)t^n$$

$$P_0(q) = 1$$

The recurrence for $P_n(q)$

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=1}^{\infty} P_{n-1}(q)t^n + \sum_{n=2}^{\infty} q^{n-1}P_{n-2}(q)t^n$$

$$P_0(q) = 1 \qquad P_1(q) = 1$$

The recurrence for $P_n(q)$

$$\sum_{n=0}^{\infty} P_n(q)t^n = 1 + \sum_{n=1}^{\infty} P_{n-1}(q)t^n + \sum_{n=2}^{\infty} q^{n-1} P_{n-2}(q)t^n$$

$$P_0(q) = 1 \qquad P_1(q) = 1$$

$$P_n(q) = P_{n-1}(q) + q^{n-1} P_{n-2}(q), n > 1$$

A *q*-Fibonacci sequence

$$P_0(q) = 1 \qquad P_1(q) = 1$$

$$P_n(q) = P_{n-1}(q) + q^{n-1} P_{n-2}(q), n > 1$$

Back to *f*(*t*)

$$f(t) = \sum_{j=0}^{\infty} \frac{t^{2j} q^{j^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^{j-1})}$$

- By the *q*–binomial series,

$$f(t) = \sum_{j=0}^{\infty} t^{2j} q^{j^2} \sum_{k=0}^{\infty} \begin{bmatrix} j+k \\ j \end{bmatrix}_q t^k$$

- By the q–binomial series,

$$f(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^{2j+k} q^{j^2} \begin{bmatrix} j+k \\ j \end{bmatrix}_q$$

$$f(t) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} t^n q^{j^2} \begin{bmatrix} j + (n - 2j) \\ j \end{bmatrix}_q$$

$$f(t) = \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\infty} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q$$

A formula for $P_n(q)$

$$f(t) = \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\infty} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q$$

$$P_n(q) = \sum_{j \geq 0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q$$

A formula for $P_n(q)$

$$P_n(q) = \sum_{j\geq 0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \xrightarrow{n\to\infty} \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\ldots(1-q^j)}$$

Another formula for $P_n(q)$

- Using the recurrence and initial conditions . . .

A q-Fibonacci sequence

$$P_0(q) = 1 \qquad P_1(q) = 1$$

$$P_n(q) = P_{n-1}(q) + q^{n-1}P_{n-2}(q), n > 1$$

$$P_0(q) = 1$$
$$P_1(q) = 1$$
$$P_2(q) = 1 + q$$
$$P_3(q) = 1 + q + q^2$$
$$P_4(q) = 1 + q + q^2 + q^3 + q^4$$
$$P_5(q) = 1 + q + q^2 + q^3 + 2q^4 + q^5 + q^6$$

$$P_0(q) = 1$$

$$P_2(q) = 1 + q$$

$$P_4(q) = 1 + q + q^2 + q^3 + q^4$$

$$P_0(q) = 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q$$

$$P_2(q) = 1 + q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$$

$$P_4(q) = 1 + q + q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - q^2 \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q$$

$$P_1(q) = 1$$

$$P_3(q) = 1 + q + q^2$$

$$P_5(q) = 1 + q + q^2 + q^3 + 2q^4 + q^5 + q^6$$

$$P_1(q) = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q$$

$$P_3(q) = 1 + q + q^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$$

$$P_5(q) = 1 + q + q^2 + q^3 + 2q^4 + q^5 + q^6 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q - (q^2 + q^3)\begin{bmatrix} 5 \\ 5 \end{bmatrix}_q$$

Another formula for $P_n(q)$

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[\begin{matrix} n \\ \left\lfloor \frac{n+5j+1}{2} \right\rfloor \end{matrix} \right]_q$$

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$$\xrightarrow{n\to\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \frac{1}{(1-q)(1-q^2)(1-q^3)\dots}$$

Another formula for $P_n(q)$

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[\begin{matrix} n \\ \left\lfloor \frac{n+5j+1}{2} \right\rfloor \end{matrix} \right]_q$$

$$\xrightarrow{n\to\infty} \prod_{m=1}^{\infty} \frac{1}{1-q^m} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}$$

Jacobi's triple product identity

- If $z \neq 0$ and $|w| < 1$, then

$$\sum_{j=-\infty}^{\infty} (-1)^j z^j w^{j^2} = \prod_{m=1}^{\infty} (1 - w^{2m})(1 - zw^{2m-1})(1 - z^{-1} w^{2m-1})$$

Jacobi's triple product identity

- Set $w = q^{5/2}$ and $z = q^{1/2}$:

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} = \prod_{m=1}^{\infty} (1-q^{5m})(1-q^{5m-2})(1-q^{5m-3})$$

Another formula for $P_n(q)$

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[\begin{matrix} n \\ \left\lfloor \frac{n+5j+1}{2} \right\rfloor \end{matrix} \right]_q$$

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$$\xrightarrow{n\to\infty} \prod_{m=1}^{\infty} \frac{(1-q^{5m})(1-q^{5m-2})(1-q^{5m-3})}{1-q^m}$$

Another formula for $P_n(q)$

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[\begin{matrix} n \\ \left\lfloor \frac{n+5j+1}{2} \right\rfloor \end{matrix} \right]_q$$

$$\xrightarrow{n\to\infty} \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})}$$

Another formula for $P_n(q)$

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \left\lfloor \frac{n+5j+1}{2} \right\rfloor \end{bmatrix}_q$$

$$\xrightarrow{n\to\infty} \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}$$