# A Generalization of the Euler-Glaisher Bijection

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joint work with
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# PARTITION definition

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- Each summand $\lambda_i$ is called a *part* of the partition $\lambda$.
- Often a canonical ordering of parts is imposed:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r.$$

# The Partitions of 6

$6 \qquad 5+1 \qquad 4+2 \qquad 4+1+1 \qquad 3+3 \qquad 3+2+1$

$3+1+1+1 \qquad 2+2+2 \qquad 2+2+1+1 \qquad 2+1+1+1+1$

$$1+1+1+1+1+1$$

# Euler’s partition identity

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# Euler's partition identity—Example

- Of the eleven partitions of $6$, four of them have only odd parts:

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- and four of them have distinct parts:

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# Frequency Notation for Partitions

- Any partition $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r$ may be written in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \ldots,$$

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$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + f_4 \cdot 4 + \ldots,$$

- or more briefly, as

$$\{f_1, f_2, f_3, f_4, \ldots\},$$

where $f_i$ represents the number of appearances of the positive integer $i$ in the partition.

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For example, the partition

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$$= 2\cdot 1 + 4\cdot 2 + 1\cdot 3 + 2\cdot 4 + 0\cdot 5 + 4\cdot 6 + 0\cdot 7 + 0\cdot 8 + 0\cdot 9$$

may be represented by the frequency sequence

$$\{2,4,1,2,0,4,0,0,0,0,0,0,\dots\}.$$

Thus each sequence $\{f_i\}_{i=1}^{\infty}$, where each $f_i$ is a nonnegative integer and only finitely many of the $f_i$ are nonzero, represents a partition of the integer $\sum_{i=1}^{\infty} i f_i$.

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- Rewrite $\lambda$ in the form

$$f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + \cdots \, .$$

- Replace each $f_i$ with its binary expansion

$$\cdots + a_{i3} \cdot 8 + a_{i2} \cdot 4 + a_{i1} \cdot 2 + a_{i0} \cdot 1.$$

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So,

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$$\begin{aligned}
& f_1 \cdot 1 + f_3 \cdot 3 + f_5 \cdot 5 + f_7 \cdot 7 + \cdots \\
= \quad & (\cdots + a_{1,3} \cdot 8 + a_{1,2} \cdot 4 + a_{1,1} \cdot 2 + a_{1,0} \cdot 1) \cdot 1 \\
+ & (\cdots + a_{3,3} \cdot 8 + a_{3,2} \cdot 4 + a_{3,1} \cdot 2 + a_{3,0} \cdot 1) \cdot 3 \\
+ & (\cdots + a_{5,3} \cdot 8 + a_{5,2} \cdot 4 + a_{5,1} \cdot 2 + a_{5,0} \cdot 1) \cdot 5 \\
& \vdots
\end{aligned}$$

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where each $a_{i,j} \in \{0, 1\}$.

# Euler's partition identity

The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into distinct parts.

# Euler's partition identity

The number of partitions of $n$ into nonmultiples of $2$ equals the number of partitions of $n$ where no part appears more than once.

# Glaisher's partition identity

The number of partitions of $n$ into nonmultiples of $m$ equals the number of partitions of $n$ where no part appears more than $m - 1$ times.

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where $f_i$=0 if $m \mid i$.

# Proof of Glaisher's identity

- Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition $\lambda$ of some positive integer $n$ into $r$ nonmultiples of $m$.
- Rewrite $\lambda$ in the form

$$f_1 \cdot 1 + f_2 \cdot 2 + f_3 \cdot 3 + \cdots ,$$

where $f_i$=0 if $m \mid i$.

- Replace each $f_i$ with its base $m$ expansion

$$\cdots + a_{i3} \cdot m^3 + a_{i2} \cdot m^2 + a_{i1} \cdot m + a_{i0} \cdot 1.$$

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So,

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= \quad & (\cdots + a_{1,3} \cdot m^3 + a_{1,2} \cdot m^2 + a_{1,1} \cdot m + a_{1,0} \cdot 1) \cdot 1 \\
+ & (\cdots + a_{2,3} \cdot m^3 + a_{2,2} \cdot m^2 + a_{2,1} \cdot m + a_{2,0} \cdot 1) \cdot 2 \\
+ & (\cdots + a_{3,3} \cdot m^3 + a_{3,2} \cdot m^2 + a_{3,1} \cdot m + a_{3,0} \cdot 1) \cdot 3 \\
& \vdots
\end{aligned}
$$

where each $0 \leq a_{i,j} \leq m - 1$.

# Partition Ideals

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**Informal Definition** A partition ideal $C$ is a set of partitions such that for each $\lambda \in C$, if one or more parts is removed from $\lambda$, the resulting partition is also in $C$.

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# The First Rogers-Ramanujan Identity

- Let $R_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$.
- Let $R_2(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_r$ of $n$ such that $\lambda_i - \lambda_{i+1} \geq 2$.
- Then $R_1(n) = R_2(n)$ for all $n$.

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- The partitions enumerated by $R_1(n)$ are those for which $f_i = 0$ whenever $i \not\equiv \pm 1 \pmod 5$.
- The partitions enumerated by $R_2(n)$ are those for which $f_i + f_{i+1} \leq 1$.

# Minimal Bounding Sequence

- Let the sequence $\{d_1^C, d_2^C, d_3^C, \ldots\}$ be defined by

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$$d_j^C = \sup_{\{f_i\}_{i=1}^{\infty} \in C} f_j,$$

- each $d_i$ is a nonnegative integer or $+\infty$.

# Euler's Theorem Revisited

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- $$\{d_j^{\mathcal{O}}\}_{j=1}^{\infty} = \{\infty, 0, \infty, 0, \infty, 0, \ldots\}$$
- Let $\mathcal{D}$ denote the set of all partitions with distinct parts.

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- Let $\mathcal{D}$ denote the set of all partitions with distinct parts.
- $$\{d_j^{\mathcal{D}}\}_{j=1}^{\infty} = \{1, 1, 1, 1, 1, 1, \ldots\}.$$

# Equivalent Partition Ideals

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- Let $p(C, n)$ denote the number of partitions of an integer $n$ in the partition ideal $C$.
- We say that two partition ideals $C_1$ and $C_2$ are equivalent, and write $C_1 \sim C_2$, if $p(C_1, n) = p(C_2, n)$ for all integers $n$.

# Euler’s Partition Identity

$$\mathcal{O} \sim \mathcal{D}.$$

# The Multiset Associated with a Partition Ideal of Order 1

- Define the multiset associated with $C$, $M(C)$, as follows:

$$M(C) := \{j(d_j^C + 1) \,\big|\, j \in \mathbb{Z}_+ \text{ and } d_j^C < \infty\}.$$

# The Multiset Associated with a Partition Ideal of Order 1

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$$M(C) := \{ j(d_j^C + 1) \,\big|\, j \in \mathbb{Z}_+ \text{ and } d_j^C < \infty \}.$$

- Andrews proved $C_1 \sim C_2$ if and only if $M(C_1) = M(C_2)$.

# The Multiset Associated with a Partition Ideal of Order 1

$$M(\mathcal{O}) = M(\mathcal{D}) = \{2, 4, 6, 8, 10, 12, \dots\}$$

# Partitions of Infinity

MacMahon defined a *partition of infinity* to be a formal expression of the form

$$(g_1-1)\cdot 1+(g_2-1)\cdot g_1+(g_3-1)\cdot(g_1g_2)+(g_4-1)\cdot(g_1g_2g_3)+\ldots$$

where

- each $g_i \geq 2$

*or* for some fixed $k$,

- $g_1, g_2, g_3, \ldots, g_{k-1} > 1$,
- $g_k = \infty$, and
- $g_{k+1} = g_{k+2} = g_{k+3} = \cdots = 1$.

# Partitions of Infinity

Note that a partition of infinity may be thought of as a minimal bounding sequence for a partition ideal of order one with

$$\begin{aligned} d_1 &= g_1 - 1 \\ d_{g_1} &= g_2 - 1 \\ d_{g_1 g_2} &= g_3 - 1 \\ d_{g_1 g_2 g_3} &= g_4 - 1 \\ &\vdots \end{aligned}$$

and

$$d_i = 0 \text{ if } i \notin \{1, g_1, g_1 g_2, g_1 g_2 g_3, \ldots\}.$$

# Partitions of Infinity

If $g_i = m$ for all $i$, we have a minimal bounding sequence for the partition ideal of the “base $m$ expansion.”

All partitions of infinity have generating function

$$\frac{1}{1-q}.$$

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- and any equivalent partition ideal of order 1 $C'$,
- there exists a collection of partitions of infinity which gives rise to a "Glaisher-type bijection" from $C$ to $C'$.
- Further, there is an explicit algorithm for finding the required partitions of infinity.

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- Let $C$ denote the set of partitions into parts not equal to $2$, $9$, $10$, $12$, $18$ or $20$.
- Let $C'$ denote the set of partitions where the parts
  - $1$, $9$, and $10$ may appear at most once,
  - $3$ and $4$ may appear at most twice,
  - $2$ may appear at most four times,
  - and all other positive integers may appear without restriction.

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  - and all other positive integers may appear without restriction.
- $C \sim C'$.

# An Inelegant Partition Identity

- $$d_j^C = \begin{cases} 0 & \text{if } j \in \{2, 9, 10, 12, 18, 20\} \\ \infty & \text{otherwise.} \end{cases}$$

$$\{d_j^{C'}\}_{j=1}^{\infty} = \{1, 4, 2, 2, \infty, \infty, \infty, \infty, 1, 1, \infty, \infty, \infty, \infty, \ldots\}$$

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- Any partition of $n$ in $C$ can be written in the form

  $$n = f_1 \cdot 1 + \sum_{i=3}^{8} f_i \cdot i + f_{11} \cdot 11 + \sum_{i=13}^{17} f_i \cdot i$$

  $$+ f_{19} \cdot 19 + \sum_{i=21}^{\infty} f_i \cdot i,$$

  where each $f_i$ is a nonnegative integer.

# An Inelegant Partition Identity

- Expand $f_1$ by the partition of infinity defined by $g_{1,1} = 2$, $g_{1,2} = 5$, $g_{1,3} = 2$, $g_{1,4} = \infty$, $g_{1,k} = 1$ if $k > 4$.

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- Expand $f_3$ by the partition of infinity defined by $g_{3,1} = 3$, $g_{3,2} = 2$, $g_{3,3} = \infty$, $g_{3,k} = 1$ if $k > 3$.

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$$
\begin{aligned}
n \;=\; & \Big(a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(2\cdot 5) + a_{1,3}(2\cdot 5\cdot 2)\Big)1 \\
& + \Big(a_{2,0}(1) + a_{2,1}(3) + a_{2,2}(3\cdot 2)\Big)3 \\
& + \Big(a_{4,0}(1) + a_{4,1}(3)\Big)4 \\
& + \Big(a_{5,0}(1)\Big)5 \\
& + \Big(a_{6,0}(1)\Big)6 \\
& + \Big(a_{7,0}(1)\Big)7 \\
& + \Big(a_{8,0}(1)\Big)8 \\
& + \Big(a_{11,0}(1)\Big)11 \\
& \vdots
\end{aligned}
$$

where $0 \leq a_{j,k} \leq g_{j,k+1} - 1 = d_{j g_{j,1} g_{j,2} \cdots g_{j,k}}$.

Apply the distributive property to obtain

$$\begin{aligned} n \;=\; & \quad a_{1,0}(1) + a_{1,1}(2) + a_{1,2}(10) + a_{1,4}(20) \\ & + a_{2,0}(3) + a_{2,1}(9) + a_{2,2}(18) \\ & + a_{4,0}(4) + a_{4,1}(12) \\ & + a_{5,0}(5) \\ & + a_{6,0}(6) \\ & \vdots \end{aligned}$$

where, in particular,

$$a_{1,0} \leq 1, \quad a_{1,1} \leq 4, \quad a_{1,2} \leq 2, \quad a_{2,0} \leq 2,$$

$$a_{2,1} \leq 1, \quad a_{4,0} \leq 2.$$

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- This $C$ maps bijectively to $C'$ via the Glaisher-type bijection where the role of the base $m$ expansion is played by an appropriate partition of infinity.
- If $C' \sim C''$ and $M(C') = M(C'')$ has no repeated elements, there is a canonical bijection between them.

# Further Canonical Bijections for Equivalent Partition Ideals

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