A formula for the partition function that “counts”

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Joint work with

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$p(100) = 190{,}569{,}192$
$p(200) = 3{,}972{,}999{,}029{,}388$
$p(500) = 2{,}300{,}165{,}032{,}574{,}323{,}995{,}027$

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So, $p_3(6)=7$

Generating Functions

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$$GF\left[\left\{\frac{1}{n!}\right\}_{n=0}^{\infty}; x\right] = e^x$$

L. Euler (1707-1783)

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)$$

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$$\begin{aligned} =\ & (1 + x + x^2 + x^3 + x^4 + \cdots) \\ & \times (1 + x^2 + x^4 + x^6 + x^8 + \cdots) \\ & \times (1 + x^3 + x^6 + x^9 + x^{12} + \cdots) \end{aligned}$$

$$
\begin{aligned}
&\left(\frac{1}{1-x^1}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\\
= \quad &(1 + x^1 + x^{1+1} + x^{1+1+1} + x^{1+1+1+1} + \cdots)\\
&\times (1 + x^2 + x^{2+2} + x^{2+2+2} + x^{2+2+2+2} + \cdots)\\
&\times (1 + x^3 + x^{3+3} + x^{3+3+3} + x^{3+3+3+3} + \cdots)
\end{aligned}
$$

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&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\
=&1+x^1+(x^2+x^{1+1})+(x^3+x^{2+1}+x^{1+1+1})+\cdots
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=\;&1+x^1+(x^2+x^{1+1})+(x^3+x^{2+1}+x^{1+1+1})+\cdots\\
=\;&\sum_{n=0}^{\infty}p_3(n)x^n = GF[p_3(n);x].
\end{aligned}
$$

$$GF[p_N(n); x] = \sum_{n=0}^{\infty} p_N(n) x^n = \prod_{j=1}^{N} \frac{1}{1 - x^j}$$

Euler's generating function for $p(n)$

$$GF[p(n); x] = \sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

Euler's pentagonal number theorem

$$(1-x)(1-x^2)(1-x^3)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\cdots .$$

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$$\sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k-1)/2}\sum_{n=0}^{\infty}p(n)x^n = 1.$$

Euler's partition recurrence

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots$$

G. H. Hardy (1877–1947)
S. Ramanujan (1887–1920)

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- $p(n) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz$ where C is any positively oriented, simple closed contour enclosing the origin and inside the unit circle.

Hardy & Ramanujan's first asymptotic formula for $p(n)$

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \text{ as } n \to \infty$$

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P. A. MacMahon (1854–1929)

Hardy & Ramanujan's "exact" asymptotic formula for $p(n)$

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\alpha\sqrt{n}} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \omega_{h,k} e^{-2\pi i h n/k} \frac{d}{dn} \left(\frac{e^{\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}}}{\sqrt{n-\frac{1}{24}}} \right) + O(n^{-1/4}),$$

with α an arbitrary constant and $\omega_{h,k}$ a certain complex $24k$th root of unity.

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$f(n) = O(g(n))$ means that $|f(n)/g(n)|$ is bounded for large enough n.

Hans Rademacher (1892–1969)

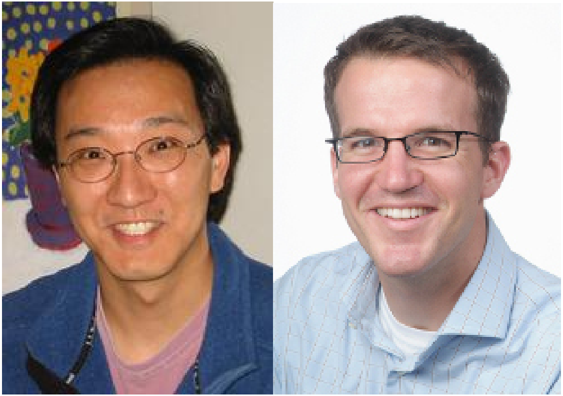
Rademacher's convergent series for $p(n)$

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \times \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \omega_{h,k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right).$$

Calculating $p(1)$ with Rademacher's series

$k = 1$ term: 1.133558447
$k = 2$ term: -0.130590021
$k = 3$ term: -0.0297786023
$k = 4$ term: 0.0157484033
$k = 5$ term: 0
$k = 6$ term: 0.0143908980

Ken Ono and Jan Bruinier



Bruinier–Ono formula for $p(n)$

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where

$$\mathrm{Tr}(n) := \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q).$$

Discriminant $-24n+1 = b^2 - 4ac$ positive definite integral binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ with the property $6 \mid a$. The group $\Gamma_0(6)$ acts on such forms, and let $\mathcal{Q}_n$ be any set of representatives of those equivalence classes with $a > 0$ and $b \equiv 1 \pmod{12}$. For each $Q(x,y)$, we let α_Q be the CM point in the upper half of the complex plane, for which $Q(\alpha_Q, 1) = 0$. Also,

$$P(z) := -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi \Im(z)}\right) F(z)$$

where

$$F(z) := \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{2\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2}$$

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with the convention that $q := e^{2\pi iz}$.

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$$\mathcal{Q}_1 = \{Q_1, Q_2, Q_3\}$$
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The CM points are

$$\alpha_{Q_1} = \frac{-1 + \sqrt{-23}}{12}, \alpha_{Q_2} = \frac{-13 + \sqrt{-23}}{24}, \alpha_{Q_3} = \frac{-25 + \sqrt{-23}}{36}.$$

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$$P(\alpha_{Q_1}) \approx 13.965486281$$
$$P(\alpha_{Q_2}) \approx 4.517256859 - 3.097890591i$$
$$P(\alpha_{Q_3}) \approx 4.517256859 + 3.097890591i$$

Ferrers graph

$$5 + 4 + 3 + 3 + 1 + 1 + 1 + 1$$

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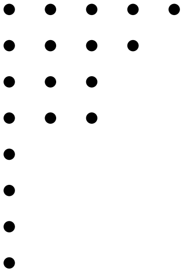
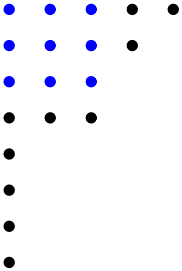


Ferrers graph with Durfee Square

$5 + 4 + 3 + 3 + 1 + 1 + 1 + 1$



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$$p(n) = \sum_{k=0}^{\lfloor\sqrt{n}\rfloor} D(n,k).$$

$$D(n,k) = \sum_{m_k=0}^{U_k} \sum_{m_{k-1}=0}^{U_{k-1}} \cdots \sum_{m_2=0}^{U_2} \left(1 + n - k^2 - \sum_{h=2}^{k} h m_h\right) \prod_{i=2}^{k} (m_i + 1),$$

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where

$$U_j := U_j(n,k) = \lfloor \frac{n - k^2 - \sum_{h=j+1}^{k} h m_h}{j} \rfloor.$$

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$$\frac{x}{(1-x)^2} = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

$$\frac{x^4}{(1-x)^2(1-x^2)^2} = \frac{1/4}{(1-x)^4} - \frac{3/4}{(1-x)^3} + \frac{11/16}{(1-x)^2} - \frac{1/8}{1-x} + \frac{1/16}{(1+x)^2} - \frac{1/8}{1+x}$$

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etc., expand RHS as a series and extract coefficient of x^n.

$$D(n,1) = n,$$

$$D(n,2) = \frac{(n-1)(2n^2-4n-3)}{48} + (-1)^n \frac{n-1}{16},$$

$$D(n,3) = \frac{(n-3)(6n^4-72n^3+184n^2+192n-235)}{25920} - (-1)^n \frac{n-3}{64} + \frac{(\omega^n+\omega^{-n})(n-3)+\left(\frac{n}{3}\right)}{81}$$

(where $\omega := e^{2\pi i/3}$ and $\left(\frac{n}{3}\right)$ is the Legendre symbol)

$$\tilde{D}(n,1) = n,$$

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Approximate $p(n)$ by

$$p_D(n) := \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \tilde{D}(n,k)$$

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and by the $k = 1$ term of the Hardy–Ramanujan–Rademacher formula:

$$p_R(n) := \frac{\cosh\left(\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{2\sqrt{3}\left(n-\frac{1}{24}\right)} - \frac{\sinh\left(\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{2\pi\sqrt{2}\left(n-\frac{1}{24}\right)^{3/2}}.$$

n	$p(n)$	$p_D(n) - p(n)$	$p_R(n) - p(n)$	$p_R(n) - p_D(n)$
5	7	**0.25**	0.26210	0.01210
10	42	−0.37905	−**0.37221**	0.00684
15	176	**0.39120**	0.56047	0.16927
20	627	−1.24394	−**1.24232**	0.00162
25	1958	2.10036	**2.09834**	−0.00202
30	5604	−3.72589	−**3.72044**	0.00545
40	37,338	−7.39250	−**7.39081**	0.00170
50	204,226	−**14.9227**	−14.9235	−0.00080
60	966,467	−**33.6090**	−33.6385	−0.02946
75	8,118,264	**79.2210**	79.2222	0.00129
100	1.9×10^8	−347.2173	−**347.2167**	0.00069
150	4.1×10^{10}	−4253.1144	−**4253.1138**	0.00058
200	4.0×10^{12}	−36202.1049	−**36202.1042**	0.00062
300	9.2×10^{15}	−1442614.889	−**1442614.887**	0.00168
500	2.3×10^{21}	−**560997650.0056**	−560997650.0066	−0.00093

In progress

Rademacher showed that

$$|p(n) - p_R(n)| < \frac{2\pi^2}{9\sqrt{3}} e^{\pi\sqrt{n/6}}.$$

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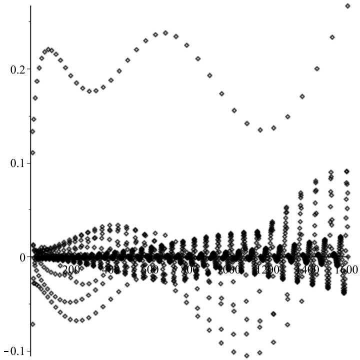
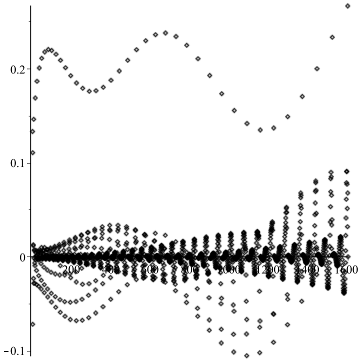
Conjecture:

$$|p(n) - p_D(n)| < \frac{2\left(\frac{n}{4}\right)^{\nu-1}}{\nu\,\Gamma(\nu+1)\Gamma(\frac{\nu}{2}+1)\Gamma(\frac{\nu}{2})}$$

where $\nu = \nu(n) = \frac{27}{50}(2+\sqrt{n})$.



$p_R(n) - p_D(n)$ for $1 \leq n \leq 1600$.



$p_R(n) - p_D(n)$ for $1 \leq n \leq 1600$.

Can we find a tight bound and an explanation for the small size of $|p_D(n) - p_R(n)|$?

Thank you for listening!