SYLVESTER'S WAVE THEORY OF PARTITIONS

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3+1
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so, we have five partions of 4.

Let $p(n)$ denote the number of partitions of n.

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$p(5) = 7$
$p(6) = 11$
$p(7) = 15$
$p(8) = 22$
$p(9) = 30$
$p(10) = 42$

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$p(6) = 11$
$p(7) = 15$
$p(8) = 22$
$p(9) = 30$
$p(10) = 42$
$p(100) = 190,569,192$
$p(200) = 3,972,999,029,388$

G. H. Hardy (1877-1947) and S. Ramanujan (1887-1920)


G. H. Hardy and S. Ramanujan (1918)

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}}, \quad \text{as} \quad n \to \infty.$$

Hardy and Ramanujan (1918)

$$p(n)$$
$$= \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\alpha\sqrt{n}} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ \gcd(h,k)=1}} \omega_{h,k} e^{-2\pi i h n/k} \frac{d}{dn} \left(\frac{\exp\left(\frac{\pi}{k} \sqrt{\frac{2}{3}(n - \frac{1}{24})} \right)}{\sqrt{n - \frac{1}{24}}} \right)$$
$$+ O(n^{-1/4}),$$

with α an arbitrary constant and $\omega_{h,k}$ a certain complex $24k$th root of unity.

Hans Rademacher (1892-1969)

H. Rademacher (1937)

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right),$$

where

$$A_k(n) = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i (s(h,k) - 2nh/k)},$$

is a Kloosterman sum,

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is a Kloosterman sum, and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right)$$

is a Dedekind sum.

Ken Ono and Jan Bruinier


In 2011, Ken Ono and Jan Bruinier announced a new formula that expresses $p(n)$ as a finite sum of certain algebraic numbers.

Let $p(n, m)$ denote the number of partitions of n into parts $\leq m$.

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So, $p(6, 3) = 7$

L. Euler (1707-1783)

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)$$

$$= (1 + x + x^2 + x^3 + x^4 + \cdots)$$

$$\times (1 + x^2 + x^4 + x^6 + x^8 + \cdots)$$

$$\times (1 + x^3 + x^6 + x^9 + x^{12} + \cdots)$$

$$\left(\frac{1}{1-x^1}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)$$
$$= (1 + x^1 + x^{1+1} + x^{1+1+1} + x^{1+1+1+1} + \cdots)$$
$$\times (1 + x^2 + x^{2+2} + x^{2+2+2} + x^{2+2+2+2} + \cdots)$$
$$\times (1 + x^3 + x^{3+3} + x^{3+3+3} + x^{3+3+3+3} + \cdots)$$

$$
\begin{aligned}
&\left(\frac{1}{1-x^1}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\\
= \;&(1+x^1+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots)\\
&\times(1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots)\\
&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\
=&1+x^1+(x^2+x^{1+1})+(x^3+x^{2+1}+x^{1+1+1})+\cdots
\end{aligned}
$$

$$\begin{aligned}
&\left(\frac{1}{1-x^1}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\\
=&\ \ (1+x^1+x^{1+1}+x^{1+1+1}+x^{1+1+1+1}+\cdots)\\
&\times(1+x^2+x^{2+2}+x^{2+2+2}+x^{2+2+2+2}+\cdots)\\
&\times(1+x^3+x^{3+3}+x^{3+3+3}+x^{3+3+3+3}+\cdots)\\
=&1+x^1+(x^2+x^{1+1})+(x^3+x^{2+1}+x^{1+1+1})+\cdots\\
=&\sum_{n=0}^{\infty}p(n,3)x^n.
\end{aligned}$$

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

Let $x = e^{2\pi i\tau}$.

$$\frac{x^{1/24}}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n)x^n,$$

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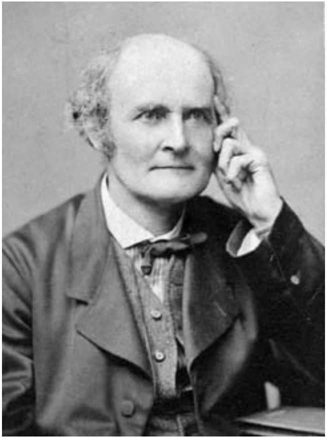
$$\frac{x^{1/24}}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n) x^n,$$

provided

$$\Im \tau > 0, \text{ i.e. } |x| < 1.$$

Arthur Cayley



$$\sum_{n=0}^{\infty} p(n,3)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{17/72}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/6}{(1-x)^3} + \frac{1/8}{1+x} + \frac{\frac{2}{9} + \frac{x}{9}}{1+x+x^2}$$

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$$= \frac{17}{72}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)x^n + \frac{1}{6}\binom{n+2}{2}x^n + \frac{1}{8}\sum_{n=0}^{\infty}\mathrm{cr}_2(1,-1)x^n$$

$$+ \frac{2}{9}\sum_{n=0}^{\infty}\mathrm{cr}_3(1,-1,0)x^n + \frac{1}{9}\sum_{n=1}^{\infty}\mathrm{cr}_3(0,1,-1)x^n$$

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$$= \sum_{n=0}^{\infty} \left(\frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \frac{1}{8}\mathrm{cr}_2(1,-1) + \mathrm{cr}_3(\frac{2}{9},-\frac{1}{9},-\frac{1}{9}) \right) x^n$$

$$p(n,3) = \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_2\left(\frac{1}{8}, -\frac{1}{8}\right) + \mathrm{cr}_3\left(\frac{2}{9}, -\frac{1}{9}, -\frac{1}{9}\right)$$

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$$= \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_6\left(\frac{25}{72}, -\frac{17}{72}, \frac{1}{72}, \frac{7}{72}, \frac{17}{72}, -\frac{17}{72}\right)$$

$$
\begin{aligned}
& p(n,3) \\
&= \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_2\left(\frac{1}{8}, -\frac{1}{8}\right) + \mathrm{cr}_3\left(\frac{2}{9}, -\frac{1}{9}, -\frac{1}{9}\right) \\
&= \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_6\left(\frac{25}{72}, -\frac{17}{72}, \frac{1}{72}, \frac{7}{72}, \frac{17}{72}, -\frac{17}{72}\right) \\
&= \text{nearest integer to } \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2}
\end{aligned}
$$

J. J. Sylvester (1814-1897)

Sylvester's theorem

$$p(n, m) = \sum_{q=1}^{m} W_{q,m},$$

where

$$W_{q,m} = \operatorname*{Res}_{x=0} \sum_{\rho \text{ prim } q\text{th root of } 1} \frac{\rho^n e^{nx}}{(1-\rho^{-1}e^{-x})(1-\rho^{-2}e^{-2x})\cdots(1-\rho^{-m}e^{-mx})}$$

$$
\begin{aligned}
W_{1,m} = & \operatorname*{Res}_{x=0} \frac{e^{nx}}{(1-e^{-x})(1-e^{-2x})\cdots(1-e^{-mx})} \\
= & \operatorname*{Res}_{x=0} \frac{e^{x(n+\frac{1}{2}(1+2+3+\cdots+m))}}{(e^{\frac{1}{2}x}-e^{-\frac{1}{2}x})(e^{\frac{1}{2}2x}-e^{-\frac{1}{2}2x})\cdots(e^{\frac{1}{2}mx}-e^{-\frac{1}{2}mx})} \\
= & \operatorname*{Res}_{x=0} \frac{e^{\nu x}}{\prod_{j=1}^{m} 2\sinh(\frac{1}{2}jx)}, \text{where } \nu = n + \tfrac{m(m+1)}{4}.
\end{aligned}
$$

- $W_{1,1} = 1$

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where $s_r = 1^r + 2^r + 3^r + \cdots + m^r$.

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- $W_{1,4} = \frac{1}{4!}\left(\frac{\nu^3}{3!} - \frac{s_2}{24}\nu\right)$
- $W_{1,5} = \frac{1}{5!}\left(\frac{\nu^4}{4!} - \frac{s_2}{24}\frac{\nu^2}{2!} + \frac{1}{576}\left(\frac{s_2^2}{2} + \frac{s_4}{5}\right)\right)$

where $s_r = 1^r + 2^r + 3^r + \cdots + m^r$.

$$W_{1,m} = \frac{1}{m!}\left(\frac{\nu^{m-1}}{(m-1)!} - J_1\frac{\nu^{m-3}}{(m-3)!} + J_2\frac{\nu^{m-5}}{(m-5)!} - \cdots\right),$$

where

$$J_1 = \frac{s_2}{24}$$

$$J_2 = \frac{1}{576}\left(\frac{s_2^2}{2} + \frac{s_4}{5}\right)$$

$$J_3 = \frac{1}{1728}\left(\frac{s_2^2}{48} + \frac{s_2 s_4}{40} + \frac{s_6}{105}\right)$$

$$\vdots$$

J. W. L. Glaisher (1848-1928)

Glaisher calculated explicit formulas for $W_{i,m}$ where $i = 1, 2, 3, 4, 5, 6$.

Research project

- Program Shalosh to find explicit formulas for $W_{i,m}$ where $i = 1, 2, 3, 4, 5, 6, 7, 8, \ldots$.

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Research project

- Program Shalosh to find explicit formulas for $W_{i,m}$ where $i = 1, 2, 3, 4, 5, 6, 7, 8, \ldots$.
- Find an explicit formula for $W_{i,m}$ for arbitrary m.
- $p(n) = p(n, n)$, so $p(n) = \sum_{i=1}^{m} W_{i,n}$.

Numerical calculations show that for reasonable size n,

$$p(n) = p(n, n) \approx W_{1,n}.$$

Research project

Find $r = r(n)$ so that

$$p(n) = p(n,n) = \text{ nearest integer to } \sum_{i=1}^{r} W_{i,n}.$$

Theory of “q-partial fractions."

Augustine O. Munagi

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n,3)x^n =& \frac{1}{(1-x)(1-x^2)(1-x^3)} \\
=& \frac{17/72}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/6}{(1-x)^3} + \frac{1/8}{1+x} + \frac{\frac{2}{9} + \frac{x}{9}}{1+x+x^2}
\end{aligned}
$$

$$\sum_{n=0}^{\infty} p(n,3)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)}$$

$$= \frac{17/72}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/6}{(1-x)^3} + \frac{1/8}{1+x} + \frac{\frac{2}{9} + \frac{x}{9}}{1+x+x^2}$$

$$= \frac{0}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/6}{(1-x)^3} + \frac{1/4}{1-x^2} + \frac{1/3}{1-x^3}$$

$$\sum_{n=0}^{\infty} p(n,3)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)}$$

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$$= \frac{0}{1-x} + \frac{1/4}{(1-x)^2} + \frac{1/6}{(1-x)^3} + \frac{1/4}{1-x^2} + \frac{1/3}{1-x^3}$$

$$= \sum_{n=0}^{\infty} \left(\frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \frac{1}{4}\mathrm{cr}_2(1,0) + \frac{1}{3}\mathrm{cr}_3(1,0,0) \right) x^n$$

$$p(n,3) = \frac{17}{72} + \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_2\left(\frac{1}{8}, -\frac{1}{8}\right) + \mathrm{cr}_3\left(\frac{2}{9}, -\frac{1}{9}, -\frac{1}{9}\right)$$

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\begin{aligned}
& p(n,3) \\
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&= \frac{n+1}{4} + \frac{1}{6}\binom{n+2}{2} + \mathrm{cr}_2\left(\frac{1}{4}, 0\right) + \mathrm{cr}_3\left(\frac{1}{3}, 0, 0\right)
\end{aligned}
$$

Research project

Find the q-partial fraction analog of Sylvester's wave theory.

Back to the beginning

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3+1 and 1+3
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2+1+1 and 1+2+1 and 1+1+2
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so, we have eight compositions of 4.

Let $c(n)$ denote the number of compositions of n.

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$c(1) = 1$

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$$c(6) = 32$$
$$c(7) = 64$$
$$\vdots$$
$$c(n) = 2^{n-1}$$

P. A. MacMahon (1854-1929)

The graph of a composition

The *MacMahon graph* of the composition $2+4+3+3$ of $n=12$

$$- \; - \; \cdot \; - \; - \; - \; - \; \cdot \; - \; - \; - \; \cdot \; - \; - \; -$$

The graph of a composition

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$$- - \cdot - - - - \cdot - - - \cdot - - -$$

A composition of n with m parts has a MacMahon graph with n dashes and $m - 1$ dots

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A composition of n with m parts has a MacMahon graph with n dashes and $m - 1$ dots , and is in one-to-one correspondence with a bit sequence of length $n - 1$ containing exactly $m - 1$ ones.

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$$2+4+3+3 \leftrightarrow (01000100100)$$

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$$2+4+3+3 \leftrightarrow (01000100100)$$

Thus $c(n) = 2^{n-1}$.

Research project

Use information about $c(n)$ to derive a formula for $p(n)$.