Identities of the Ramanujan-Slater Type

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Rogers and Ramanujan

The Rogers-Ramanujan identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 1 \pmod 5}} \frac{1}{1-q^j}$$

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$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2 \pmod 5}} \frac{1}{1-q^j}$$

Integer Partitions

A *partition* λ of the integer n into ℓ parts is an ℓ-tuple

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

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The quantity n is called the *weight* of λ and is denoted $|\lambda|$.

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The quantity n is called the *weight* of λ and is denoted $|\lambda|$.
The number of parts of λ is called the *length* $\ell(\lambda)$ of λ.

There are seven partitions of 5:

$$(5) \quad (4,1) \quad (3,2) \quad (3,1,1) \quad (2,2,1)$$
$$(2,1,1,1) \quad (1,1,1,1,1)$$

1st RR identity—combinatorial version

The number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n where

$$\lambda_i - \lambda_{i+1} \geqq 2$$

for $i = 1, 2, \ldots, \ell - 1$

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for $i = 1, 2, \dots, \ell - 1$ equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

Example: For $n = 10$, the relevant partitions are

$$(10) \quad (9,1) \quad (8,2) \quad (7,3) \quad (6,4) \quad (6,3,1)$$

$$(9,1) \quad (6,4) \quad (6,1,1,1,1) \quad (4,4,1,1) \quad (4,1,1,1,1,1,1)$$
$$(1,1,1,1,1,1,1,1,1,1)$$

a-generalized Rogers-Ramanujan identities

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)}$$

$$= \prod_{k\geq 1} \frac{1}{1-aq^k} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j-1)/2}(1-aq^{2j})}{1-a} \prod_{i=1}^{j} \frac{1-aq^{i-1}}{1-q^i}$$

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)}$$

$$= \prod_{k\geq 1} \frac{1}{1-aq^k} \sum_{j=0}^{\infty} (-1)^j a^{2j} q^{j(5j+3)/2}(1-aq^{2j+1}) \prod_{i=1}^{j} \frac{1-aq^i}{1-q^i}.$$

a-generalized Rogers-Ramanujan identities

$$F_2(a) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)}$$

$$F_1(a) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)}$$

associated q-difference equations

$$F_1(a) = F_2(aq)$$

$$F_2(a) = F_1(a) + aqF_2(aq)$$

Combinatorial a-generalized 1st RR

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} R(\ell,n) a^{\ell} q^n,$$

where $R(\ell, n)$ denotes the number of partitions λ of length ℓ and weight n in which $\lambda_i - \lambda_{i+1} \geqq 2$ for $1 \leqq i \leqq \ell - 1$.

Ramanujan's lost notebook

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1-q^3)(1-q^9)\cdots(1-q^{6j-3})}{(1-q)^2(1-q^3)^2\cdots(1-q^{2j-1})^2(1-q^4)(1-q^8)\dots(1-q^{4j})}$$

$$= \frac{1+q^2+q^6+\cdots}{1-q-q^3+\cdots}(1-q^6)(1-q^{18})\dots$$

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$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-q)(1-q^3)\cdots(1-q^{2j-1})\times(1-q^4)(1-q^8)\dots(1-q^{4j})}$$

$$= \prod_{\substack{j\geqq 1 \\ j \text{ odd or } j\equiv\pm 2 \pmod{12}}} \frac{1}{1-q^j}$$

a-generalized Ramanujan identity

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-aq)(1-aq^3)\cdots(1-aq^{2j-1})\times(1-q^4)(1-q^8)\ldots(1-q^{4j})}$$

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$$= \prod_{j=1}^{\infty} \frac{1+aq^{4j-2}+a^2q^{8j-4}}{1-aq^{2j-1}}$$

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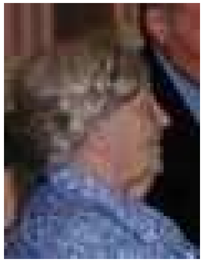
where $s(\ell,n)$ denotes the number of partitions of weight n and length ℓ where even parts may appear at most twice and are not multiples of 4.

Ramanujan-Slater Mod 8 Identity

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\cdots(1+q^{2j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} = \prod_{\substack{j\geqq 1 \\ j\equiv \pm 1,4 \pmod 8}} \frac{1}{1-q^j}$$

First Göllnitz-Gordon partition identity

The number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n where

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equals the number of partitions of n into parts congruent to $\pm 1, 4 \pmod 8$.

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\cdots(1+q^{2j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})}$$

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$$= \sum_{j=0}^{\infty} \frac{q^{j^2} q(q^{-1}+1)q^3(q^{-3}+1)\cdots q^{2j-1}(q^{-(2j+1)}+1)}{(1-q^2)(1-q^4)\cdots(1-q^{2j})}$$

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Signed Partitions

A *signed partition* σ of the integer n is a partition pair (π, ν) where $|\pi| - |\nu| = n$.

- The positive parts of σ are the parts of π.
- The negative parts of σ are the parts of ν.

Andrews' G-G for Signed Partitions

The number of signed partitions $\sigma = (\pi, \nu)$ of n where

- all positive parts are even and at least $2\ell(\pi)$, and
- all negative parts are odd, distinct and less than $2\ell(\pi)$

equals the number of (ordinary) partitions of n into parts congruent to $\pm 1, 4 \pmod 8$.

Example: For $n = 10$, the relevant partitions are

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$$\Big((10),\emptyset\Big) \quad \Big((6,4),\emptyset\Big) \quad \Big((10,4),(3,1)\Big) \quad \Big((8,6),(3,1)\Big)$$
$$\Big((6,6,6),(5,3)\Big)$$

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$$\big((6,6,6),(5,3)\big)$$

$$(9,1) \quad (7,1,1,1) \quad (4,4,1,1) \quad (4,1,1,1,1,1,1)$$
$$(1,1,1,1,1,1,1,1,1,1)$$

Ramanujan 's lost notebook

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1-q^3)(1-q^9)\cdots(1-q^{6j-3})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})\times(1-q)(1-q^3)\ldots(1-q^{2j-1})}$$

$$= \frac{1-q-q^5+q^8+\cdots}{1-q-q^2+q^5+\cdots}(1-q^9)(1-q^{27})\cdots$$

Ramanujan 's lost notebook

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$$= \frac{1-q-q^5+q^8+\cdots}{1-q-q^2+q^5+\cdots}(1-q^9)(1-q^{27})\cdots$$

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

$$= \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}}} \frac{1}{1-q^j}$$

In 1981, Andrews gave the following partition theoretic interpretation: Let $\rho_1(n)$ denote the number of partitions of n subject to the conditions:

- no part appears more than twice,
- no odd part exceeds the number of even parts,
- among the even parts (arranged in nonincreasing size) only the second, fourth, sixth, etc., may be subsequently repeated.

Let $\rho_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$.
Then $\rho_1(n) = \rho_2(n)$ for all n.

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

$$=\sum_{j=0}^{\infty} \frac{q^{4j^2}(1+q^{-1}+q^{-2})\cdots(1+q^{-(2j-1)}+q^{-(4j-2)})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

The number of signed partitions $\sigma = (\pi, \nu)$ of n where

- $\ell(\pi)$ is even, and each positive part is even and $\geqq \ell(\pi)$,
- the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice

equals the number of partitions of n in parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$.

Mod 18 Identities—Loxton

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}(1+q^3)(1+q^6)\cdots(1+q^{3n-3})}{(1+q)(1+q^2)\cdots(1+q^{n-1})\times(1-q)(1-q^2)\cdots(1-q^{2n})}$$

$$=\prod_{\substack{j\geqq 1\\ j\equiv\pm2,\pm3,\pm4,\pm5,\pm6 \pmod{18}}}\frac{1}{1-q^j}$$

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Mod 18 Identities—McLaughlin-S.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(1+q^3)(1+q^6)\cdots(1+q^{3n})}{(1+q)(1+q^2)\cdots(1+q^n)\times(1-q)(1-q^2)\cdots(1-q^{2n+1})}$$

$$= \prod_{j \geqq 1} \frac{(1-q^{18j-3})(1-q^{18j-15})}{(1-q^{3j-1})(1-q^{3j-2})}$$

Mod 18 Identities—McLaughlin-S.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(1+q^3)(1+q^6)\cdots(1+q^{3n}) \times (1-q^{n+1})}{(1+q)(1+q^2)\cdots(1+q^n) \times (1-q)(1-q^2)\cdots(1-q^{2n+1})} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2, \pm 3, \pm 6, \pm 7, \pm 8 \pmod{18}}} \frac{1}{1-q^j}$$

Rogers-Ramanujan and Lie Algebras

- Let $\mathfrak{g}$ be be affine Kac-Moody Lie Algebra $A_1^{(1)}$ or $A_2^{(2)}$.
- Let $\{h_0, h_1\}$ be the usual basis of a maximal toral subalgebra T of $\mathfrak{g}$.

- For all dominant integral $\lambda \in \tilde{T}^*$, there is an essentially unique irreductible, integrable, highest weight module $L(\lambda)$ called the *standard module*, assuming WLOG that $\lambda(d) = 0$.
- $\lambda = s_0\Lambda_0 + s_1\Lambda_1$ where Λ_0 and Λ_1 are the fundamental weights, given by

$$\Lambda_i(h_j) = \delta_{ij} \text{ and } \Lambda_i(d) = 0.$$

- There is an infinite product $F_{\mathfrak{g}}$ associated with $\mathfrak{g}$ called the “fudge factor."
- $\mathfrak{g}$ has a certain infinite-dimensional Heisenberg subalgebra known as the “principal Heisenberg vacuum subalgebra" $\mathfrak{s}$.

In the case of $A_1^{(1)}$ for standard modules of odd level $2k+1$,

$$\chi(\Omega((2k-i+2)\Lambda_0+(i-1)\Lambda_1))$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+3)j})(1-q^{(2k+3)j-i})(1-q^{(2k+3)j-2k-3+i})}{1-q^j}$$

$$=\sum_{n_1,n_2,\ldots,n_k\geqq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}},$$

where $1\leqq i\leqq k+1$ and $N_j:=n_j+n_{j+1}+\cdots+n_k$.

In the case of $A_1^{(1)}$ for standard modules of even level $2k$,

$$\chi(\Omega((2k-i+1)\Lambda_0+(i-1)\Lambda_1))$$
$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+2)j})(1-q^{(2k+2)j-i})(1-q^{(2k+2)j-2k-2+i})}{1-q^j}$$
$$=\sum_{n_1,n_2,\dots,n_k\geqq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q^2;q^2)_{n_k}},$$

where $1\leqq i\leqq k+1$ and $N_j:=n_j+n_{j+1}+\cdots+n_k$.

Define

$$Q(w,x) :=$$

$$\prod_{j \geqq 1} (1 - w^j)(1 + xw^{j-1}) \left(1 + \frac{w^j}{x}\right) \left(1 - \frac{w^{2j-1}}{x^2}\right) (1 - x^2 w^{2j-1}).$$

The principal character for the level ℓ standard modules associated with $A_2^{(2)}$ is

$$\chi((\ell - 2i + 2)\Lambda_0 + (i-1)\Lambda_1)) = \frac{Q(q^{\ell+3}, -q^i)}{\prod_{j \geqq 1}(1 - q^j)},$$

where $1 \leqq i \leqq 1 + \lfloor \frac{\ell}{2} \rfloor$.

Principal characters of level 6 standard mod

$$\chi(6\Lambda_0) = \frac{Q(q^9,-q)}{\prod_{j\geqq 1} 1-q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}}$$

$$\chi(4\Lambda_0+\Lambda_1) = \frac{Q(q^9,-q^2)}{\prod_{j\geqq 1} 1-q^j} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}}$$

$$\chi(2\Lambda_0+2\Lambda_1) = \frac{Q(q^9,-q^3)}{\prod_{j\geqq 1} 1-q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}}$$

$$\chi(4\Lambda_1) = \frac{Q(q^9,-q^4)}{\prod_{j\geqq 1} 1-q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n}{(q^2;q^2)_n(q^{n+2};q)_{n+1}}$$