On the Rogers-Selberg Identities and Gordon's Theorem

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Rogers and Ramanujan

Rogers-Ramanujan Identities

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$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2 \pmod 5}} \frac{1}{1-q^j}$$

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Analytic Disclaimer

Assume throughout that $|q| < 1$.

Rising q-factorial notation

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Rogers and Selberg

Partitions

- A *partition* π of an integer n is a nonincreasing finite sequence of positive integers

$$\pi = \{\pi_1, \pi_2, \pi_3, \ldots, \pi_s\}$$

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- The seven partitions of 5 are thus

$$\{5\} \qquad \{4,1\} \qquad \{3,2\} \qquad \{3,1,1\}$$

$$\{2,2,1\} \qquad \{2,1,1,1\} \qquad \{1,1,1,1,1\}.$$

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- The *multiplicity* of the integer j in the partition π, denoted $m_j(\pi)$, is the number of times j appears in π.

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$$\langle 5 \rangle \qquad \langle 1\ 4 \rangle \qquad \langle 2\ 3 \rangle \qquad \langle 1^2 3 \rangle$$

$$\langle 1\ 2^2 \rangle \qquad \langle 1^3 2 \rangle \qquad \langle 1^5 \rangle.$$

Euler

Euler's partition identity (1748)

The number of partitions of n into parts which differ by at least 1
equals the number of partitions of n into parts congruent to $\pm 1 \pmod 4$

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MacMahon and Schur

 

Combinatorial RR (1917)

The number of partitions of n into parts which differ by at least 2
equals the number of partitions of n into parts congruent to $\pm 1 \pmod{5}$

Schur's partition identity (1926)

The number of partitions of n into parts which differ by at least 3 and where no consecutive multiples of 3 appear equals the number of partitions of n into parts congruent to $\pm 1 \pmod 6$

Lehmer and Alder

Lehmer's nonexistence theorem (1946)

The number of partitions of n into parts which differ by at least d
is *not* equal to the number of partitions of n into parts taken from *any* set of integers whatsoever if $d > 2$.

Alder's nonexistence theorem (1948)

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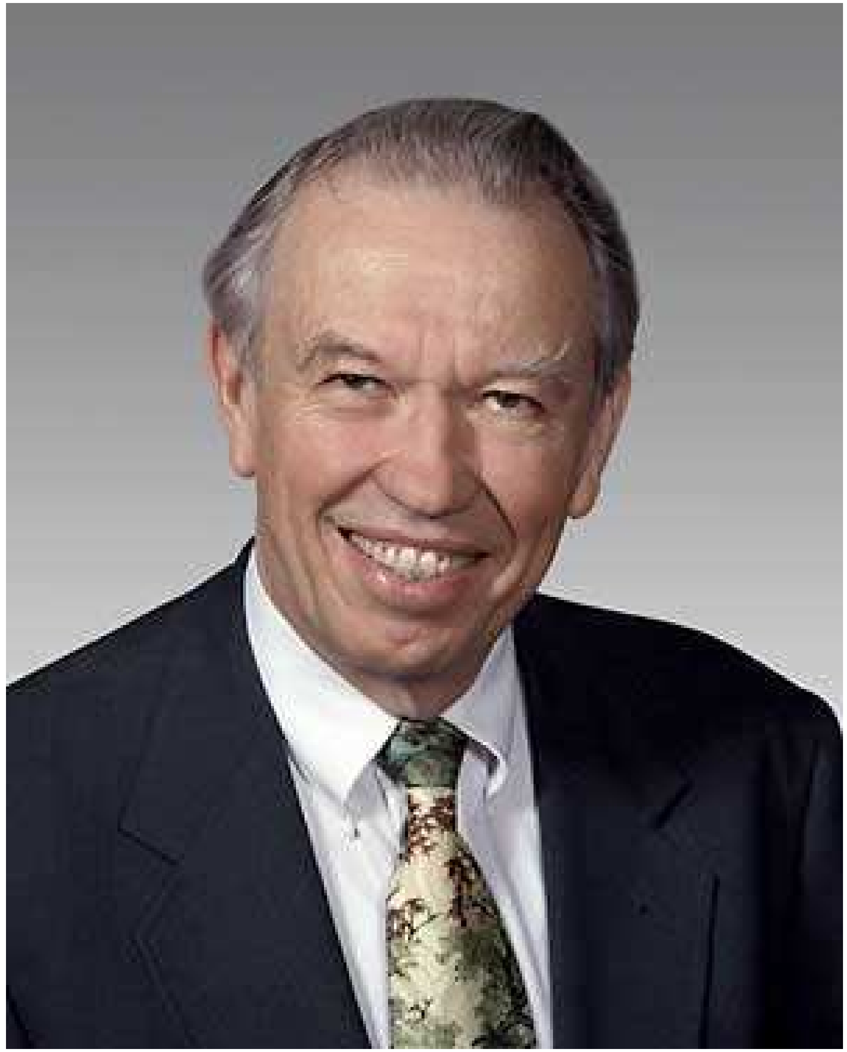
Then $G_{k,i}(n) = C_{k,i}(n)$ for $1 \leqq i \leqq k$ and all integers n.

George E. Andrews



Andrews-Gordon Identity (1974)

$$\sum_{n_1,n_2,\ldots,n_{k-1}\geqq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}}$$

$$= \prod_{\substack{j=1 \\ j\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty} \frac{1}{1-q^j},$$

where $N_j := n_j + n_{j+1} + \cdots + n_{k-1}$.

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$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{q^{(n_1+n_2)^2+n_2^2+n_2}}{(q;q)_{n_1}(q;q)_{n_2}} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 2 \pmod 7}} \frac{1}{1-q^j}.$$

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$$\sum_{j=0}^{\infty}\frac{q^{2j^2+2j}(-q^{2j+1};q)_{\infty}}{(q^2;q^2)_j} = \prod_{\substack{j\geqq 1\\ j\not\equiv 0,\pm 2 \pmod 7}}\frac{1}{1-q^j}.$$

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Then $A(n) = C_{3,2}(n)$ for all n.

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- Let $\mathcal{A}$ denote the set of partitions enumerated by $A(n)$ in Andrews' combinatorial interpretation of the 2nd Rogers-Selberg identity.

Characterization of $\mathcal{G}$ partitions

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- no number appears more than twice as a part,
- if r appears twice, then neither $r-1$ nor $r+1$ appear, and
- 1 appears at most once.

Characterization of $\mathcal{A}$ partitions

A partition $\pi \in \mathcal{A}$ may be thought of as a union of two partitions :

- a partition into 2's, 4's, 6's, $\ldots$, $2j$'s with all parts repeated, and

Characterization of $\mathcal{A}$ partitions

A partition $\pi \in \mathcal{A}$ may be thought of as a union of two partitions :

- a partition into 2's, 4's, 6's, . . . , $2j$'s with all parts repeated, and
- a partition into distinct parts greater than $2j$.

Example

$$\{30, 27, 19, 15, 15, 13, 12, 10, 8, 8, 4, 4, 2, 1\} \in \mathcal{G}$$

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$$\downarrow$$

$$\{30, 27, 19, 15, 14, 12, 8, 7\} \cup \langle 2^4 4^4 6^2 \rangle \in \mathcal{A}$$

$$\{30, 27, 19, 15, 15, 13, 12, 10, 8, 8, 4, 4, 2, 1\}$$

$$\begin{array}{cccccccccccccccccc}
30 & 27 & 19 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & + & 13 & 12 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & & & \\
 & & & & & + & 2 & 2 & 2 & 2 & & & & & & & & \\
 & & & & & + & 2 & 1 & & & & & & & & & & \\
\hline
30 & 27 & 19 & 15 & 14 & 12 & 8 & 7 & 6 & 6 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccccc} 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 30 & 27 & 19 & 15 & 14 & 12 & 8 & 7 & 6 & 6 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\ & & & \uparrow & & & & & & & & & & & & & & \end{array}$$

So first row is

$$30 \quad 27 \quad 19 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2$$

$$\begin{array}{ccccccccccc} 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 13 & 12 & 10 & 6 & 5 & 4 & 4 & 2 & 2 & 2 & 2 \\ & & & \uparrow & & & & & & & \end{array}$$

So first row and second rows are

$$\begin{array}{cccccccccccccccccccc} 30 & 27 & 19 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ & & & 13 & 12 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & & & \end{array}$$

$$\begin{array}{cccc} 4 & 3 & 2 & 1 \\ \hline 4 & 3 & 2 & 2 \\ \uparrow & & & \end{array}$$

So the first three rows are

$$\begin{array}{cccccccccccccccccc} 30 & 27 & 19 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ & & & 13 & 12 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & & & \\ & & & & & & 2 & 2 & 2 & 2 & & & & & & & & \end{array}$$

$$
\begin{array}{cc}
2 & 1 \\
\hline
2 & 1 \\
\uparrow &
\end{array}
$$

$$
\begin{array}{cccccccccccccccccc}
30 & 27 & 19 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
 & & + & 13 & 12 & 10 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & & & \\
 & & & & & + & 2 & 2 & 2 & 2 & & & & & & & & \\
 & & & & & + & 2 & 1 & & & & & & & & & & \\
\hline
30 & 27 & 19 & 15 & 14 & 12 & 8 & 7 & 6 & 6 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2
\end{array}
$$

Analogous interpretations of the other two Rogers-Selberg identities can be given and they in turn can be mapped similarly to the $i = 1$ and $i = 3$ instances of the partitions enumerated by the $k = 3$ case of Gordon's theorem.

"A partition bijection related to the Rogers-Selberg identities and Gordon’s theorem,"
Journal of Combinatorial Theory, Series A
115/1 (2008) 67-83.

http://math.georgiasouthern.edu/~asills

Thank you!