Rogers–Ramanujan type identities

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L. Euler (1707-1783)
S. Ramanujan (1887–1920)
L. J. Rogers (1862–1933)
\[
\sum_{n \geq 0} \frac{q^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{m=1}^{\infty} \frac{1}{1 - q^m} \quad \text{(Euler)}
\]
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\sum_{n \geq 0} \frac{q^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{m=1}^{\infty} \frac{1}{1 - q^m} \quad \text{(Euler)}
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\sum_{n \geq 0} \frac{q^{n^2}}{(1 - q)^2(1 - q^2)^2 \cdots (1 - q^n)^2} = \prod_{m \geq 1} \frac{1}{1 - q^m} \quad \text{(Euler)}
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\[
\sum_{n \geq 0} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)} = \prod_{m \geq 1} \frac{1}{1 - q^m} \quad m \equiv 1 \text{(mod 5)}
\tag{Rogers}
\]
Rising $q$-factorial notation

$$(a)_n = (a; q)_n := \prod_{j=0}^{n-1} (1 - a q^j),$$
Rising $q$-factorial notation

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$$(a)_{\infty} = (a; q)_{\infty} := \prod_{j \geq 0} (1 - aq^j),$$
Rising $q$-factorial notation

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\[(a)_{\infty} = (a; q)_{\infty} := \prod_{j \geq 0} (1 - aq^j),\]

\[(a_1, a_2, \ldots, a_r; q)_{\infty} := \prod_{j=1}^{r} (a_j; q)_{\infty}.\]
For $|ab| < 1$, 

$$f(a, b) := \sum_{n \in \mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}.$$
Ramanujan’s “theta” function

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\[ f(a, b) := \sum_{n \in \mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}. \]

Jacobi’s triple product identity

\[ f(a, b) = (a, b, ab; ab)_\infty. \]
Ramanujan’s notation

\[ f(-q) := f(-q, -q^2) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(3n-1)/2} = (q)_{\infty} \]
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\[ \psi(-q) := f(-q, -q^3) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(2n-1)} = \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \]
Rogers–Ramanujan identities

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty}.
\]

\[
\sum_{n \geq 0} \frac{q^{n(n+1)}}{(q)_n} = \frac{f(-q, -q^4)}{(q)_\infty}.
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Rogers–Ramanujan identities

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Ramanujan really enjoyed identities of this type. Over 50 are recorded in the lost notebook.
If \((\alpha_n(a, q), \beta_n(a, q))\) satisfies

\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},
\]

then \((\alpha_n, \beta_n)\) is called a Bailey pair with respect to \(a\),
Bailey pairs, Bailey’s lemma

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\]

then \((\alpha_n, \beta_n)\) is called a Bailey pair with respect to \(a\), and 

\((\alpha'_n(a, q), \beta'_n(a, q))\) is also a Bailey pair, where

\[
\alpha'_r(a, q) = \frac{(\rho_1)_r (\rho_2)_r}{(aq/\rho_1)_r (aq/\rho_2)_r} \left( \frac{aq}{\rho_1 \rho_2} \right)^r \alpha_r
\]

and

\[
\beta'_n(a, q) = \sum_{j=0}^{n} \frac{(\rho_1)_j (\rho_2)_j (aq/\rho_1 \rho_2)_{n-j}}{(aq/\rho_1)_n (aq/\rho_2)_n (q)_{n-j}} \left( \frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q).
\]
Limiting cases of Bailey’s lemma

\[
\sum_{n \geq 0} q^{n^2} \beta_n(1, q) = \frac{1}{f(-q)} \sum_{r \geq 0} q^{r^2} \alpha_r(1, q) \quad \text{(PBL)}
\]

\[
\sum_{n \geq 0} q^{n^2} (-q; q^2)_n \beta_n(1, q^2) = \frac{1}{\psi(-q)} \sum_{r \geq 0} q^{r^2} \alpha_r(1, q^2) \quad \text{(HBL)}
\]

\[
\sum_{n \geq 0} q^{n(n+1)/2} (-1)^n \beta_n(1, q) = \frac{2}{\varphi(-q)} \sum_{r \geq 0} \frac{q^{r(r+1)/2}}{1 + q^r} \alpha_r(1, q) \quad \text{(SBL)}
\]
In the 1940’s, Bailey found a number of examples of Bailey pairs, and used them to generate RR type identities.
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Freeman Dyson contributed a number of RR type identities to Bailey’s papers.

Lucy Slater found many Bailey pairs, and used them to generate a list of 130 RR type identities.
Letting

$$[d \mid n] = \begin{cases} 1 & \text{if } d \mid n \\ 0 & \text{if } d \nmid n \end{cases},$$

we define

$$\alpha_n^{(d,e,k)}(a, q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1 - a^{1/e})(q^{d/e}; q^{d/e})_{n/d}} \times \frac{(1 - a^{1/e} q^{2n/e})(a^{1/e}; q^{d/e})_{n/d}[d \mid n]}{[d \mid n]},$$
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\]

\[
\tilde{\alpha}_n^{(d,e,k)}(a, q) := q^{n(d-n)/2de} a^{-n/de} \frac{(-a^{1/e}; q^{d/e})_{n/d}}{(-q^{d/e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q),
\]
General Bailey pairs

Letting

\[ [d \mid n] = \begin{cases} 
1 & \text{if } d \mid n \\
0 & \text{if } d \nmid n 
\end{cases} , \]

we define

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\alpha_n^{(d,e,k)}(a, q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1 - a^{1/e})(q^{d/e}; q^{d/e})_{n/d}} \times (1 - a^{1/e} q^{2n/e})(a^{1/e}; q^{d/e})_{n/d} [d \mid n],
\]

\[
\tilde{\alpha}_n^{(d,e,k)}(a, q) := q^{n(d-n)/2de} a^{-n/d e} \frac{(-a^{1/e}; q^{d/e})_{n/d}}{(-q^{d/e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q),
\]

\[
\tilde{\alpha}_n^{(d,e,k)}(a, q) := (-1)^{n/d} q^{n^2/2de} \frac{(q^{d/2e}; q^{d/e})_{n/d}}{(a^{1/e} q^{d/2e}; q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a, q).
\]
For any positive integer triples $(d, e, k)$, upon inserting any of these $\alpha$’s into any of the limiting cases of Bailey’s lemma with $a = 1$, the resulting series is summable via Jacobi’s triple product identity.
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For certain \((d, e, k)\), the resulting expression for \(\beta\) is a very well-poised \(6\phi_5\), summable by a theorem of F. H. Jackson.

Using only this, and an associated families of \(q\)-difference equations, one can recover the majority of Slater’s list, as well as other identities.
The Bailey pair that arises from

\[
\left( \alpha_n^{(1,1,2)}(a, q), \beta_n^{(1,1,2)}(a, q) \right)
= \left( \frac{(-1)^n a^n q^{n(3n-1)/2} (1 - aq^{2n}) (a)_n}{(1 - a)(q)_n}, \frac{1}{(q)_n} \right)
\]

yields
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yields

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty} \quad \text{upon insertion into (PBL),}
\]
The Bailey pair that arises from

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\left( \alpha_{n}^{(1,1,2)}(a, q), \beta_{n}^{(1,1,2)}(a, q) \right)
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yields

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\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty} \quad \text{upon insertion into (PBL),}
\]
- \[
\sum_{n \geq 0} \frac{q^{n(n+1)}(-1)_n}{(q)_n} = \frac{\varphi(-q^2)}{\varphi(-q)} \quad \text{upon insertion into (SBL), and}
\]
The Bailey pair that arises from

\[
\left( \alpha_n^{(1,1,2)}(a, q), \beta_n^{(1,1,2)}(a, q) \right)
= \left( \frac{(-1)^n a^n q^{n(3n-1)/2} (1 - aq^{2n})(a)_n}{(1 - a)(q)_n}, \frac{1}{(q)_n} \right)
\]

yields

- \[ \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} = \frac{f(-q^2, -q^3)}{(q)_\infty} \] upon insertion into (PBL),
- \[ \sum_{n \geq 0} \frac{q^{n(n+1)}(-1)_n}{(q)_n} = \frac{\varphi(-q^2)}{\varphi(-q)} \] upon insertion into (SBL), and
- \[ \sum_{n \geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{f(-q^3, -q^5)}{\psi(-q)} \] upon insertion into (HBL).
Ramanujan’s pre-discovery of the Göllnitz–Gordon identities

In Slater, we find

$$
\sum_{n \geq 0} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{f(-q^3, -q^4)}{\psi(-q)},
$$  \hspace{1cm} (S. 36)

$$
\sum_{n \geq 0} \frac{q^{n(n+2)}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{f(-q, -q^7)}{\psi(-q)}.
$$  \hspace{1cm} (S. 34)

Replace $q$ by $-q$ to get the analytic Göllnitz–Gordon identities.
Ramanujan’s pre-discovery of the Göllnitz–Gordon identities

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In the lost notebook, we find
\[\sum_{n \geq 0} (-1)^n q^{n^2} (q; q^2)_n = \frac{\psi(q^4)}{f(q, q^7)}, \quad (RLN II: Ent 1.7.11)\]
\[\sum_{n \geq 0} (-1)^n q^{n(n+2)} (q; q^2)_n = \frac{\psi(q^4)}{f(q^3, q^5)}. \quad (RLN II: Ent 1.7.12)\]
Ramanujan’s pre-discovery of the Göllnitz–Gordon identities

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\]

Replace \( q \) by \( -q \) to get the analytic Göllnitz–Gordon identities.
A family of mod 24 identities

\[\sum_{n \geq 0} q^{n(n+2)} (-q; q^2)_n (-1; q^6)_n = \frac{f(-q, -q^{11})f(-q^{10}, -q^{14})}{\psi(-q)(q^{24}; q^{24})_\infty} \]  
(M.-S.)

\[\sum_{n=0}^{\infty} q^{n^2} (-q^3; q^6)_n = \frac{f(-q^2, -q^{10})f(-q^8, -q^{16})}{\psi(-q)(q^{24}; q^{24})_\infty} \]  
(RLN II: Ent 5.3.9)

\[\sum_{n \geq 0} q^{n^2} (-q; q^2)_n (-1; q^6)_n = \frac{f(-q^3, -q^9)f(-q^6, -q^{18})}{\psi(-q)(q^{24}; q^{24})_\infty} \]  
(M.-S.)

\[\sum_{n \geq 0} q^n(-q^3; q^6)_n \frac{(q^2; q^2)_{2n}(1 - q^{2n+1})}{(q^2; q^2)_{2n}(1 - q^{2n+1})} = \frac{f(-q^4, -q^8)f(-q^4, -q^{20})}{\psi(-q)(q^{24}; q^{24})_\infty} \]  
(M.-S.)

\[\sum_{n \geq 0} q^n(-q; q^2)_{n+1} (-q^6; q^6)_n = \frac{f(-q^5, -q^7)f(-q^2, -q^{22})}{\psi(-q)(q^{24}; q^{24})_\infty} \]  
(M.-S.)
Combinatorial Rogers–Ramanujan

Rogers, Ramanujan, Bailey, and Slater did not consider the combinatorial aspect of their work.
A partition $\lambda$ of $n$ is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the parts of $\lambda$) that sum to $n$. 
Rogers, Ramanujan, Bailey, and Slater did not consider the combinatorial aspect of their work.

A partition $\lambda$ of $n$ is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the parts of $\lambda$) that sum to $n$. The seven partitions of 5 are

$$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1).$$
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The number of partitions of $n$ into parts that mutually differ by at least 2 equals the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$. 
Combinatorial Rogers–Ramanujan

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A *partition* $\lambda$ of $n$ is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the *parts* of $\lambda$) that sum to $n$. The seven partitions of 5 are

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The number of partitions of $n$ into parts that mutually differ by at least 2 equals the number of partitions of $n$ into parts congruent to $\pm 1 \pmod{5}$.

The number of partitions of $n$ into parts greater than 1 that mutually differ by at least 2 equals the number of partitions of $n$ into parts congruent to $\pm 2 \pmod{5}$. 
Let $k$ be a positive integer and $1 \leq i \leq k$. 
Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts \( \not\equiv 0, \pm i \pmod{2k + 1} \).
Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k + 1}$. Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where

- at most $i - 1$ of the parts of $\lambda$ equal 1,
- $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$. 

Note: The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.
Let $k$ be a positive integer and $1 \leq i \leq k$. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where
- at most $i - 1$ of the parts of $\lambda$ equal 1,
- $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$.
Then $A_{k,i}(n) = B_{k,i}(n)$ for all $n$. 

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Let $B_{k,i}(n)$ denote the number of partitions $\lambda$ of $n$ where

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Then $A_{k,i}(n) = B_{k,i}(n)$ for all $n$.

**Note:** The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.
G. Andrews’ analytic counterpart to Gordon’s theorem

\[
\sum_{n_{k-1} \geq n_{k-2} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_{k-1}^2 + n_i + n_{i+1} + \cdots + n_{k-1}}}{(q)_{n_1} (q)_{n_2-n_1} (q)_{n_3-n_2} \cdots (q)_{n_{k-1}-n_{k-2}}} = f(-q^i, -q^{2k+1-i}) \frac{1}{(q)_{\infty}}.
\]
In the 1980’s J. Lepowsky and R. Wilson showed that the principally specialized characters of standard modules for the odd levels of $A_1^{(1)}$ are given by the The Andrews–Gordon identity.
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The two Rogers–Ramanujan identities occur at level 3.

The even levels of $A_{1}^{(1)}$ correspond to D. Bressoud’s even modulus analog of Andrews–Gordon.
The Rogers–Ramanujan identities also occur at level 2 of $A_2^{(2)}$. 

Capparelli’s identities (1988)
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Performing an analogous analysis of the level 3 modules of $A_2^{(2)}$, S. Capparelli discovered:

The number of partitions of $n$ into parts $\equiv \pm 2, \pm 3 \pmod{12}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ where

- $\lambda_i - \lambda_{i+1} \geq 2$,
- $\lambda_i - \lambda_{i+1} = 2 \implies \lambda_i \equiv 1 \pmod{3}$,
- $\lambda_i - \lambda_{i+1} = 3 \implies \lambda_i \equiv 0 \pmod{3}$
In an analogous study of the level 4 modules of \( A_2^{(2)} \), D. Nandi conjectured three partition identities, one of which is:

The number of partitions of \( n \) into parts \( \equiv \pm 2, \pm 3, \pm 4 \pmod{14} \) equals the number of partitions \( (\lambda_1, \lambda_2, \ldots, \lambda_l) \) of \( n \) where \( \lambda_i - \lambda_i + 1 \geq 2 \lambda_i - \lambda_i + 2 \geq 3 \lambda_i - \lambda_i + 2 = 3 \Rightarrow \lambda_i \neq \lambda_i + 1, \lambda_i - \lambda_i + 2 = 4\) and \( 2 \nmid \lambda_i = \Rightarrow \lambda_i \neq \lambda_i + 2 \).

Consider the first differences \( \Delta \lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_l - 1 - \lambda_l) \). None of the following subwords are permitted in \( \Delta \lambda \):

\((3,3,0), (3,2,3,0), (3,2,2,3,0), \ldots, (3,2,2,2,2,2,\ldots,2,3,0)\).
Nandi’s identities (2014)

In an analogous study of the level 4 modules of $A^{(2)}_2$, D. Nandi conjectured three partition identities, one of which is:

The number of partitions of $n$ into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ where

- $\lambda_i - \lambda_{i+1} \geq 2$
- $\lambda_i - \lambda_{i+2} \geq 3$
- $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$
- $\lambda_i - \lambda_{i+2} = 3$ and $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$
- $\lambda_i - \lambda_{i+2} = 4$ and $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$

Consider the first differences

$\Delta \lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{l-1} - \lambda_l)$. None of the following subwords are permitted in $\Delta \lambda$:

$(3, 3, 0), (3, 2, 3, 0), (3, 2, 2, 3, 0), \ldots, (3, 2, 2, 2, 2, \ldots, 2, 3, 0)$. 

Drew Sills

Rogers–Ramanujan type identities
Related to level 3 standard modules of \(D_4^{(3)}\), Kandade and Russell conjectured several partition identities, including:
Related to level 3 standard modules of $D^{(3)}_4$, Kandade and Russell conjectured several partition identities, including:

The number of partitions of $n$ into parts $\equiv \pm 1, \pm 3 \pmod{9}$ equals the number of partitions $\lambda$ of $n$ such that

- $\lambda_j - \lambda_{j+2} \geq 3$,
- $\lambda_j - \lambda_{j+1} \leq 1 \implies 3 \mid (\lambda_j + \lambda_{j+1})$. 
“A framework of Rogers–Ramanujan identities and their arithmetic properties”, Duke Math. J., to appear. Griffin, Ono, and Warnaar find a framework which extends the RR identities to doubly infinite families of $q$ series identities. For $a = 1$ or $2$ and $m, n \geq 1$,

$$\sum_{\lambda_{1} < m} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^n) = \text{an infinite product modular function},$$

where $P_{\lambda}(x_1, x_2, \ldots; q)$ are Hall–Littlewood polynomials. These $q$-series are specialized characters of affine Kac–Moody algebras.
Thank you for listening!