Rogers–Ramanujan type identities

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L. Euler (1707-1783)
S. Ramanujan (1887–1920)
L. J. Rogers (1862–1933)



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(Rogers)

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$$(a_1, a_2, \dots a_r; q)_\infty := \prod_{j=1}^{r} (a_j; q)_\infty.$$

Ramanujan's "theta" function

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$$f(a,b) := \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}.$$

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Jacobi's triple product identity

$$f(a, b) = (a, b, ab; ab)_\infty.$$

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$$\psi(-q) := f(-q,-q^3) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(2n-1)} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$$

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Over 50 are recorded in the lost notebook.

Bailey pairs, Bailey's lemma

If $(\alpha_n(a,q), \beta_n(a,q))$ satisfies

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

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then (α_n, β_n) is called a *Bailey pair with respect to a*, and $(\alpha_n'(a,q), \beta_n'(a,q))$ is also a Bailey pair, where

$$\alpha_r'(a,q) = \frac{(\rho_1)_r(\rho_2)_r}{(aq/\rho_1)_r(aq/\rho_2)_r}\left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r$$

and

$$\beta_n'(a,q) = \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}}{(aq/\rho_1)_n(aq/\rho_2)_n(q)_{n-j}}\left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j(a,q).$$

Limiting cases of Bailey's lemma

$$\sum_{n\geq 0} q^{n^2}\beta_n(1,q) = \frac{1}{f(-q)}\sum_{r\geq 0} q^{r^2}\alpha_r(1,q) \qquad \text{(PBL)}$$

$$\sum_{n\geq 0} q^{n^2}(-q;q^2)_n\beta_n(1,q^2) = \frac{1}{\psi(-q)}\sum_{r\geq 0} q^{r^2}\alpha_r(1,q^2) \qquad \text{(HBL)}$$

$$\sum_{n\geq 0} q^{n(n+1)/2}(-1)_n\beta_n(1,q) = \frac{2}{\varphi(-q)}\sum_{r\geq 0} \frac{q^{r(r+1)/2}}{1+q^r}\alpha_r(1,q) \qquad \text{(SBL)}$$

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- Freeman Dyson contributed a number of RR type identities to Bailey's papers.
- Lucy Slater found many Bailey pairs, and used them to generate a list of 130 RR type identities.

General Bailey pairs

Letting

$$[\![d \mid n]\!] = \begin{cases} 1 & \text{if } d \mid n \\ 0 & \text{if } d \nmid n \end{cases},$$

we define

$$\alpha_n^{(d,e,k)}(a,q) := \frac{(-1)^{n/d} a^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1-a^{1/e})(q^{d/e};q^{d/e})_{n/d}} \times (1-a^{1/e}q^{2n/e})(a^{1/e};q^{d/e})_{n/d}[\![d \mid n]\!],$$

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$$\tilde{\alpha}_n^{(d,e,k)}(a,q) := q^{n(d-n)/2de} a^{-n/de} \frac{(-a^{1/e};q^{d/e})_{n/d}}{(-q^{d/e};q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a,q),$$

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$$\bar{\alpha}_n^{(d,e,k)}(a,q) := (-1)^{n/d} q^{n^2/2de} \frac{(q^{d/2e};q^{d/e})_{n/d}}{(a^{1/e}q^{d/2e};q^{d/e})_{n/d}} \alpha_n^{(d,e,k)}(a,q).$$

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- For certain (d, e, k), the resulting expression for β is a very well-poised ${}_6\phi_5$, summable by a theorem of F. H. Jackson.
- Using only this, and an associated families of q-difference equations, one can recover the majority of Slater's list, as well as other identities.

The Bailey pair that arises from

$$\left(\alpha_n^{(1,1,2)}(a,q), \beta_n^{(1,1,2)}(a,q)\right) = \left(\frac{(-1)^n a^n q^{n(3n-1)/2}(1-aq^{2n})(a)_n}{(1-a)(q)_n}, \frac{1}{(q)_n} \right)$$

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- $\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{f(-q^3,-q^5)}{\psi(-q)}$ upon insertion into (HBL).

Ramanujan's pre-discovery of the Göllnitz–Gordon identities

In Slater, we find

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{f(-q^3,-q^4)}{\psi(-q)}, \qquad \text{(S. 36)}$$

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Replace q by $-q$ to get the analytic Göllnitz–Gordon identities.

A family of mod 24 identities

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{f(-q,-q^{11})f(-q^{10},-q^{14})}{\psi(-q)(q^{24};q^{24})_\infty}$$

(M.-S.)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q^3;q^6)_n}{(q^2;q^2)_{2n}} = \frac{f(-q^2,-q^{10})f(-q^8,-q^{16})}{\psi(-q)(q^{24};q^{24})_\infty}$$

(RLN II: Ent 5.3.9)

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n(-1;q^6)_n}{(q^2;q^2)_{2n}(-1;q^2)_n} = \frac{f(-q^3,-q^9)f(-q^6,-q^{18})}{\psi(-q)(q^{24};q^{24})_\infty} \quad \text{(M.-S.)}$$

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q^3;q^6)_n}{(q^2;q^2)_{2n}(1-q^{2n+1})} = \frac{f(-q^4,-q^8)f(-q^4,-q^{20})}{\psi(-q)(q^{24};q^{24})_\infty} \quad \text{(M.-S.)}$$

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_{n+1}(-q^6;q^6)_n}{(q^4;q^4)_n(q^{2n+4};q^2)_{n+1}} = \frac{f(-q^5,-q^7)f(-q^2,-q^{22})}{\psi(-q)(q^{24};q^{24})_\infty}$$

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The number of partitions of n into parts that mutually differ by at least 2 equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

The number of partitions of n into parts greater than 1 that mutually differ by at least 2 equals the number of partitions of n into parts congruent to $\pm 2 \pmod 5$.

B. Gordon's combinatorial generalization of RR (1961)

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Let $B_{k,i}(n)$ denote the number of partitions λ of n where

- at most $i-1$ of the parts of λ equal 1,
- $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$.

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Note: The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.

G. Andrews' analytic counterpart to Gordon's theorem

$$\sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+n_{i+1}+\cdots+n_{k-1}}}{(q)_{n_1}(q)_{n_2-n_1}(q)_{n_3-n_2}\cdots(q)_{n_{k-1}-n_{k-2}}}$$

$$= \frac{f(-q^i,-q^{2k+1-i})}{(q)_\infty}.$$

Connections to Lie algebras

- In the 1980's J. Lepowsky and R. Wilson showed that the principally specialized characters of standard modules for the odd levels of $A_1^{(1)}$ are given by the The Andrews–Gordon identity.

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- The even levels of $A_1^{(1)}$ correspond to D. Bressoud's even modulus analog of Andrews–Gordon.

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The number of partitions of n into parts $\equiv \pm 2, \pm 3 \pmod{12}$ equals the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of n where

- $\lambda_i - \lambda_{i+1} \geq 2$,
- $\lambda_i - \lambda_{i+1} = 2 \implies \lambda_i \equiv 1 \pmod{3}$,
- $\lambda_i - \lambda_{i+1} = 3 \implies \lambda_i \equiv 0 \pmod{3}$

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The number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of n where

- $\lambda_i - \lambda_{i+1} \geq 2$
- $\lambda_i - \lambda_{i+2} \geq 3$
- $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,
- $\lambda_i - \lambda_{i+2} = 3$ and $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$.
- $\lambda_i - \lambda_{i+2} = 4$ and $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$,
- Consider the first differences $\Delta\lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{l-1} - \lambda_l)$. None of the following subwords are permitted in $\Delta\lambda$: $(3,3,0), (3,2,3,0), (3,2,2,3,0), \dots, (3,2,2,2,2,\dots,2,3,0)$.

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Related to level 3 standard modules of $D_4^{(3)}$, Kandade and Russell conjectured several partition identities, including:

The number of partitions of *n* into parts $\equiv \pm 1, \pm 3 \pmod 9$ equals the number of partitions λ of *n* such that

- $\lambda_j - \lambda_{j+2} \geq 3$,
- $\lambda_j - \lambda_{j+1} \leq 1 \implies 3 \mid (\lambda_j + \lambda_{j+1})$.

M. Griffin, K. Ono, O. Warnaar (2014)

"A framework of Rogers–Ramanujan identities and their arithmetic properties", *Duke Math. J.*, to appear.
Griffin, Ono, and Warnaar find a framework which extends the RR identities to doubly infinite families of q series identities. For $a = 1$ or 2 and $m, n \geq 1$,

$$\sum_{\substack{\lambda \\ \lambda_1 < m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q^n) = \text{an infinite product modular function,}$$

where $P_\lambda(x_1, x_2, \dots; q)$ are Hall–Littlewood polynomials. These q-series are specialized characters of affine Kac–Moody algebras.

Thank you for listening!