Rademacher’s Infinite Partial Fractions Conjecture

Andrew Sills

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A *partition* of an integer $n$ is a representation of $n$ as a sum of positive integers where order of summands (parts) does not matter.
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A *partition* of an integer $n$ is a representation of $n$ as a sum of positive integers where order of summands (parts) does not matter.

Let $p(n)$ denote the number of partitions of $n$. Let $p_N(n)$ denote the number of partitions of $n$ into at most $N$ parts.
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Let $p(n)$ denote the number of partitions of $n$. Let $p_N(n)$ denote the number of partitions of $n$ into at most $N$ parts.
\[
\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)\cdots}.
\]
\[
\sum_{n=0}^{\infty} p_4(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}
\]
\[
\sum_{n=0}^{\infty} p_4(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}
\]

\[
= \frac{1}{(x - 1)^4(x + 1)^2(x - e^{2\pi i/3})(x - e^{4\pi i/3})(x - i)(x + i)}
\]
\[
\sum_{n=0}^{\infty} p_4(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}
\]

\[
= \frac{1}{(x - 1)^4(x + 1)^2(x - e^{2\pi i/3})(x - e^{4\pi i/3})(x - i)(x + i)}
\]

\[
= -\frac{17}{72} + \frac{59}{288} + \frac{1}{8} + \frac{1}{24} + \frac{1}{8} + \frac{1}{32}
\]

\[
\frac{1}{x - 1} + \frac{1}{(x - 1)^2} + \frac{1}{(x - 1)^3} + \frac{1}{(x - 1)^4} + \frac{1}{x + 1} + \frac{1}{(x + 1)^2}
\]

\[
+ \frac{1}{18} - \frac{\sqrt{3}}{54}
\]

\[
\frac{i}{x - e^{2\pi i/3}} + \frac{1}{18} + \frac{\sqrt{3}}{54}
\]

\[
\frac{i}{x - e^{4\pi i/3}} + \frac{-1}{16} \frac{i}{x - i} + \frac{1}{16} \frac{i}{x + i}
\]
\[ \sum_{n=0}^{\infty} p_N(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \cdots (1 - x^N)} \]
\[
\sum_{n=0}^{\infty} p_N(n) x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \cdots (1 - x^N)}
\]

\[
= (-1)^N \prod_{k=1}^{N} \prod_{0 \leq h < k \atop \gcd(h, k) = 1} \frac{1}{(x - e^{2\pi i h/k})^{\lfloor N/k \rfloor}}
\]
\[
\sum_{n=0}^{\infty} p_N(n) x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \cdots (1 - x^N)}
\]

\[
= (-1)^N \prod_{k=1}^{N} \prod_{0 \leq h < k, \gcd(h,k) = 1} \frac{1}{(x - e^{2\pi i h/k})^{\lfloor N/k \rfloor}}
\]

\[
= \sum_{k=1}^{N} \sum_{0 \leq h < k, \gcd(h,k) = 1} \sum_{l=1}^{\lfloor N/k \rfloor} \frac{C_{h,k,l}(N)}{(x - e^{2\pi i h/k})^l}
\]
\[
\sum_{n=0}^{\infty} p_4(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}
\]

\[
= \frac{1}{(x - 1)^4(x + 1)^2(x - e^{2\pi i/3})(x - e^{4\pi i/3})(x - i)(x + i)}
\]

\[
= \frac{-17}{72} + \frac{59}{288} + \frac{-1}{8} + \frac{1}{24} + \frac{1}{8} + \frac{1}{32} + \frac{1}{18} - \frac{\sqrt{3}}{54} + \frac{1}{18} + \frac{\sqrt{3}}{54} + \frac{-1}{16} + \frac{1}{16}
\]

Andrew Sills  The Rademacher Conjecture
\[
\sum_{n=0}^{\infty} \rho_4(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}
\]

\[
= \frac{1}{(x-1)^4(x+1)^2(x-e^{2\pi i/3})(x-e^{4\pi i/3})(x-i)(x+i)}
\]

\[
= \frac{C_{0,1,1}(4)}{x-1} + \frac{C_{0,1,2}(4)}{(x-1)^2} + \frac{C_{0,1,3}(4)}{(x-1)^3} + \frac{C_{0,1,4}(4)}{(x-1)^4} + \frac{C_{1,2,1}(4)}{x+1} + \frac{C_{1,2,2}(4)}{(x+1)^2}
\]

\[
+ \frac{C_{1,3,1}(4)}{x-e^{2\pi i/3}} + \frac{C_{2,3,1}(4)}{x-e^{4\pi i/3}} + \frac{C_{1,4,1}(4)}{x-i} + \frac{C_{3,4,1}(4)}{x+i}
\]
Topics in Analytic Number Theory, pp. 301–302:
“The $C_{h,k,l}(N)$ can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes soon very cumbersome with increasing $N$. 

Andrew Sills
The Rademacher Conjecture
"The $C_{h,k,l}(N)$ can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes soon very cumbersome with increasing $N$. . . . I propose the following conjecture."

*Topics in Analytic Number Theory*, pp. 301–302:
For fixed relatively prime integers $h$ and $k$ with $0 \leq h < k$, and fixed positive integer $l$,

$$\lim_{N \to \infty} C_{h,k,l}(N)$$

exists
Rademacher’s Conjecture

For fixed relatively prime integers $h$ and $k$ with $0 \leq h < k$, and fixed positive integer $l$,

$$
\lim_{N \to \infty} C_{h,k,l}(N)
$$

exists and is equal to

$$
C_{h,k,l}(\infty) = -2\pi \left( \frac{\pi}{12} \right)^{3/2} \cdot \frac{e^{\pi i (s(h,k) + 2hl/k)}}{k^{5/2}} \cdot \Delta_{\alpha}^{l-1} L_{3/2} \left( -\frac{\pi^2}{6k^2} (\alpha + 1) \right),
$$

evaluated at $\alpha = \frac{1}{24}$,
Rademacher’s Conjecture

For fixed relatively prime integers $h$ and $k$ with $0 \leq h < k$, and fixed positive integer $l$, 

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\lim_{N \to \infty} C_{h,k,l}(N)
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$$

evaluated at $\alpha = \frac{1}{24}$, where 

$$
s(h, k) = \sum_{\mu=1}^{k-1} \left( \frac{\mu}{k} - \lfloor \frac{\mu}{k} \rfloor - \frac{1}{2} \right) \left( \frac{h\mu}{k} - \lfloor \frac{h\mu}{k} \rfloor - \frac{1}{2} \right),
$$
Rademacher’s Conjecture

For fixed relatively prime integers $h$ and $k$ with $0 \leq h < k$, and fixed positive integer $l$,

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evaluated at $\alpha = \frac{1}{24}$, where

$$s(h, k) = \sum_{\mu=1}^{k-1} \left( \frac{\mu}{k} - \lfloor \frac{\mu}{k} \rfloor - \frac{1}{2} \right) \left( \frac{h \mu}{k} - \lfloor \frac{h \mu}{k} \rfloor - \frac{1}{2} \right),$$

$\Delta_{\alpha}$ is the forward difference operator, so that

$$\Delta_{\alpha}^j = \sum_{h=0}^{j} (-1)^h \binom{j}{h} f(\alpha + j - h),$$
Rademacher’s Conjecture

For fixed relatively prime integers $h$ and $k$ with $0 \leq h < k$, and fixed positive integer $l$,

\[ \lim_{N \to \infty} C_{h,k,l}(N) \]

exists and is equal to

\[ C_{h,k,l}(\infty) = -2\pi \left( \frac{\pi}{12} \right)^{3/2} e^{\pi i (s(h,k) + 2hl/k)} \frac{\Delta^{l-1}_{\alpha} L_{3/2}}{k^{5/2}} \left( -\frac{\pi^2}{6k^2 (\alpha + 1)} \right), \]

evaluated at $\alpha = \frac{1}{24}$, where

\[ s(h, k) = \sum_{\mu=1}^{k-1} \left( \frac{\mu}{k} - \lfloor \frac{\mu}{k} \rfloor - \frac{1}{2} \right) \left( \frac{h\mu}{k} - \lfloor \frac{h\mu}{k} \rfloor - \frac{1}{2} \right), \]

$\Delta_{\alpha}$ is the forward difference operator, so that

\[ \Delta^j_{\alpha} = \sum_{h=0}^j (-1)^h \binom{j}{h} f(\alpha + j - h), \]

and $L_{3/2}(-y^2) = -\frac{1}{2\sqrt{\pi}y^2} \left( 2\cos(2y) - \frac{\sin(2y)}{y} \right)$. 

Andrew Sills  The Rademacher Conjecture
(Meager) evidence in favor

\[ C_{0,1,1}(1) = -1 \]

\[ C_{0,1,1}(2) = -\frac{1}{4} = -0.25 \]

\[ C_{0,1,1}(3) = -\frac{17}{72} \approx -0.23611111111 \]

\[ C_{0,1,1}(4) = -\frac{17}{72} \approx -0.23611111111 \]

\[ C_{0,1,1}(5) = -\frac{20831}{86400} \approx -0.2410995370 \]

\[ \vdots \]

\[ C_{0,1,1}(\infty) = -\frac{6}{25} \left( 1 + \frac{2\sqrt{3}}{5\pi} \right) \approx -0.292927573960 \]
Rademacher writes, “No explicit formula for $C_{h,k,l}(N)$ is known, not even for the simplest case $h = 0$, $k = 1$, $l = 1$, and variable $N$.\"
Rademacher writes, “No explicit formula for $C_{h,k,l}(N)$ is known, not even for the simplest case $h = 0$, $k = 1$, $l = 1$, and variable $N$. In 2003, Andrews offered

$$C_{0,1,1}(N + 1) = \frac{-1}{(N + 1)!}$$

$$\times \sum_{h_1=1}^{1} \sum_{h_2=1}^{2} \cdots \sum_{h_N=1}^{N} (\rho_{i+1}^{h_i}) H_N \left( \frac{\rho_2^{h_1}}{1 - \rho_2^{h_1}}, \ldots, \frac{\rho_{N+1}^{h_N}}{1 - \rho_{N+1}^{h_N}} \right),$$

where $\rho_j = e^{2\pi i/j}$, and $H_N(x_1, \ldots, x_n)$ is the $N$th homogeneous symmetric function of $x_1, x_2, \ldots, x_n$. 

Andrew Sills

The Rademacher Conjecture
Of the preceding formula, Andrews wrote,

“It may be reasonably objected that [it] . . . provides little hope of proving Rademacher’s Conjecture. At most it may suggest the value of finding better formulas for $C_{h,k,l}(N)$.”
“It is a capital mistake to theorise before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.”
Rademacher gave $C_{0,1,1}(N)$ for $N = 1, 2, 3, 4, 5$. 

Andrew Sills

The Rademacher Conjecture
• Rademacher gave $C_{0,1,1}(N)$ for $N = 1, 2, 3, 4, 5$.
• Andrews gave $C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 8$
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Andrews gave $C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 8$.
Davidson and Gagola gave $C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 45$.
• Rademacher gave $C_{0,1,1}(N)$ for $N = 1, 2, 3, 4, 5$.
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• We found a fast recurrence and implemented in Maple allowing us to find $C_{0,1,1}(N)$ for...
Rademacher gave $C_{0,1,1}(N)$ for $N = 1, 2, 3, 4, 5$.
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We found a fast recurrence and implemented in Maple allowing us to find $C_{0,1,1}(N)$ for $N = 1, 2, 3, \ldots, 1500$. 
$C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 100$
$C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 200$
$C_{0,1,1}(N)$ is an oscillating function of $N$ of “period” 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as $N$ increases.
$C_{0,1,1}(N)$ is an oscillating function of $N$ of “period” 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as $N$ increases. For $N = 99$ through $N = 803$, the local maxima (resp. minima) occur when $N = 32n + 3$ (resp. $N = 32n + 19$).
\( C_{0,1,1}(N) \) is an oscillating function of \( N \) of “period” 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as \( N \) increases. For \( N = 99 \) through \( N = 803 \), the local maxima (resp. minima) occur when \( N = 32n + 3 \) (resp. \( N = 32n + 19 \).
\[ C_{0,1,1}(N) = \frac{e^{Nu}}{N^2} \left( \alpha \sin(\beta + Nv) + O(N^{-1}) \right), \]
\[ C_{0,1,1}(N) = \frac{e^{Nu}}{N^2} \left( \alpha \sin(\beta + Nu) + O(N^{-1}) \right) , \]

where \( \alpha \approx 5.39532, \beta \approx 1.92792, u \approx 0.0680762, \]
\( v \approx -0.196576 \).
\[ C_{0,1,1}(N) = \frac{e^{Nu}}{N^2} \left( \alpha \sin(\beta + N\nu) + O(N^{-1}) \right), \]

where \( \alpha \approx 5.39532, \beta \approx 1.92792, u \approx 0.0680762, \nu \approx -0.196576. \)

The period of oscillations is \(-2\pi/\nu \approx 31.9631.\)
Revised Conjecture

\[ C_{0,1,l}(N) \] is an oscillating function of \( N \) of “period” 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as \( N \) increases.
$C_{0,1,l}(N)$ is an oscillating function of $N$ of “period” 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as $N$ increases.

For $N < 800$, the local maxima (resp. minima) occur when $N \equiv 12 - 9l \pmod{32}$ (resp. $N \equiv 28 - 9l \pmod{32}$).
Selected values of $C_{0,1,l}(N)$

<table>
<thead>
<tr>
<th>$N \backslash l$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
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</tr>
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$\ldots$

$$\frac{(-1)^N}{N!}$$
### Selected values of $C_{0,1,i}(N)$

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\[\begin{align*}
\frac{(-1)^{N+1}}{4(N-2)!} & \quad \frac{(-1)^N}{N!}
\end{align*}\]
## Selected values of $C_{0,1,l}(N)$

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\[
\begin{align*}
\left(\frac{(-1)^N(9N^2-13N+26)}{288(N-2)!}\right) & \quad \left(\frac{(-1)^{N+1}}{4(N-2)!}\right) & \quad \left(-\frac{1}{N!}\right)
\end{align*}
\]
"Top down" conjecture for Rademacher's coefficients

\[ C_{0,1,N-r}(N) = \frac{(-1)^{N+r}}{4^r N! r!} P_{0,1,N-r}(N), \]

where, for \( r > 0 \), \( P_{0,1,N-r}(N) \) is a convex, alternating, monic polynomial of degree \( 2r \) whose only real roots are 0 and 1.
Theorem

\[ P_{0,1,N}(N) = 1 \]
Theorem

\[ P_{0,1,N}(N) = 1 \]

\[ P_{0,1,N-1}(N) = N^2 - N \]
Theorem

\[ P_{0,1,N}(N) = 1 \]

\[ P_{0,1,N-1}(N) = N^2 - N \]

\[ P_{0,1,N-2}(N) = N^4 - \frac{22}{9} N^3 + \frac{13}{3} N^2 - \frac{26}{9} N \]
Theorem

\[ P_{0,1,N}(N) = 1 \]

\[ P_{0,1,N-1}(N) = N^2 - N \]

\[ P_{0,1,N-2}(N) = N^4 - \frac{22}{9} N^3 + \frac{13}{3} N^2 - \frac{26}{9} N \]

\[ P_{0,1,N-3}(N) = N^6 - \frac{13}{3} N^5 + \frac{43}{3} N^4 - 25N^3 + \frac{98}{3} N^2 - \frac{56}{3} N \]
For $r > 0$, $P_{0,1,N-r}(N)$ is a monic polynomial of degree $2r$ with roots 0 and 1.
Theorem (Cormac O’Sullivan)

For $r > 0$, $P_{0,1,N-r}(N)$ is a monic polynomial of degree $2r$ with roots 0 and 1.

$$P_{0,1,N-r}(N) = 4^r \sum_{k=0}^{r} (-1)^k \binom{r}{k} k! \sum_{(1^{m_1}2^{m_2}\cdots k^{m_k})-k} \frac{1}{m_1! m_2! \cdots m_k!}$$

$$\times \prod_{h=1}^{m} \left( \frac{(-1)^{h-1} B_h}{h \cdot h!} (s_h(N) - N) \right)^{m_h},$$

where $B_j$ is the $j$th Bernoulli number, $[r]$ is a Sterling number of the first kind, and $s_h(N)$ is defined as:

$$s_h(N) = h^{-1} \sum_{j=0}^{h} \frac{(-1)^h}{j!} \binom{h-1}{j} (N - j).$$
Theorem (Cormac O’Sullivan)

For \( r > 0 \), \( P_{0,1,N-r}(N) \) is a monic polynomial of degree \( 2r \) with roots 0 and 1.

\[
P_{0,1,N-r}(N) = 4^r \sum_{k=0}^{r} (-1)^k \binom{r}{k} k! \sum_{(1^{m_1}2^{m_2}\cdots k^{m_k}) \vdash k} \frac{1}{m_1!m_2!\cdots m_k!} \times \prod_{h=1}^{m} \left( \frac{(-1)^{h-1}B_h}{h \cdot h!} \left( s_h(N) - N \right) \right)^{m_h},
\]

where \( B_j \) is the \( j \)th Bernoulli number,
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where $B_j$ is the $j$th Bernoulli number, $\binom{r}{k}$ is a Sterling number of the first kind, and

$$s_h(N) = \frac{1}{h+1} \sum_{j=0}^{h} \binom{h+1}{j+1} (-1)^{h-j} B_{h-j} N^{j+1}. $$
$C_{1,2,1}(N)$ for $N = 1, 2, \ldots, 150$
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It appears that \( \lim_{N \to \infty} C_{h,k,l}(N) \) does not exist for any \( h, k, l \). Nonetheless, we observed that \( C_{0,1,l}(24l) \) appears to be very close to Rademacher’s conjectured value of \( C_{0,1,l}(\infty) \).
How might we rigorously disprove RC?

Using the idea of “period 32,” it appears that $C_{0,1,1}(32n)$ is strictly increasing.
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$$\frac{C_{0,1,1}(32n + 32)}{C_{0,1,1}(32n)} > 1,$$

for all sufficiently large $n$, then we’d be done.
How might we rigorously disprove RC?

\[ \log C_{0,1,1}(32(n + 3)) \] as a function of \( n \) for \( n = 1 \ldots 23 \)
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Linear regression: $\log C_{0,1,1}(32n) = 2.0280756n - 8.20939$
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Correlation coefficient: \( r = 0.998549 \).
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\[ C_{0,1,1}(N) = \text{Res}_{x=1} \left( \sum_{n=0}^{\infty} p_N(n)x^n \right) = \text{Res}_{x=1} \frac{1}{(1 - x)(1 - x^2) \cdots (1 - x^N)} \]
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= \frac{1}{2\pi i} \int_{\gamma} \frac{dx}{(1-x) \cdots (1-x^N)},
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where \( \gamma \) is any simple, closed, positively oriented contour enclosing \( x = 1 \) but none of the other primitive \( j \)-th roots of unity for \( j = 2, 3, \ldots, N \).
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where $\gamma$ is any simple, closed, positively oriented contour enclosing $x = 1$ but none of the other primitive $j$-th roots of unity for $j = 2, 3, \ldots, N$. e.g.

$\gamma_N : |x - 1| = \frac{1}{N}$. 
How might we rigorously disprove RC?

\[ C_{0,1,1}(N) = \frac{1}{N} \int_0^1 \frac{e^{2\pi it} \, dt}{\left(1 - \left(1 + \frac{e^{2\pi it}}{N}\right)\right) \cdots \left(1 - \left(1 + \frac{e^{2\pi it}}{N}\right)^N\right)} \]
How might we rigorously disprove RC?

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\[ = 4(-1)^N N^{N-1} \int_{t=0}^{1/2} \frac{\cos(2\pi t(1 - N)) dt}{\Re \Gamma(1 + \exp(2\pi it)/N(N + 1))} \]
How might we rigorously disprove RC?

Then, estimate

\[
\frac{C_{0,1,1}(32(n + 1))}{C_{0,1,1}(32n)} = 32^{32} \left( \frac{n + 1}{n} \right)^{32n - 1} (n + 1)^{32} \times \frac{\int_{t=0}^{1/2} \cos(2\pi(32n+31)t)dt}{\int_{t=0}^{1/2} \exp(2\pi it/(32n+32)(32n+33))} \]

\[
\times \int_{t=0}^{1/2} \frac{\cos(2\pi(32n-1)t)dt}{\exp(2\pi it/(32n)(32n+1))} \]

Andrew Sills  The Rademacher Conjecture
Implications and applications of the Rademacher Coefficients

\[ \sum_{n \geq 0} \rho_m(n) x^n = \sum_{k=1}^{m} \sum_{0 \leq h < k} \sum_{\gcd(h, k) = 1}^{\lfloor m/k \rfloor} \frac{C_{h,k,l}(m)}{(x - \rho_{h,k})^l} \]
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\[ \sum_{n \geq 0} \rho_m(n) x^n = \sum_{k=1}^{m} \sum_{0 \leq h < k}^{\text{gcd}(h,k)=1} \sum_{l=1}^{\lfloor m/k \rfloor} \frac{C_{h,k,l}(m)}{(x - \rho_{h,k})^l} \]

Thus

\[ p(n) = p_n(n) = \sum_{k=1}^{n} \sum_{0 \leq h < k}^{\text{gcd}(h,k)=1} \sum_{l=1}^{\lfloor n/k \rfloor} (-1)^l \rho_{h,k}^{-l(n+1)} C_{h,k,l}(n) \binom{n + l - 1}{n}, \]

where \( \rho_{h,k} := \exp(2\pi i h/k) \).