Rademacher's Infinite Partial Fractions Conjecture

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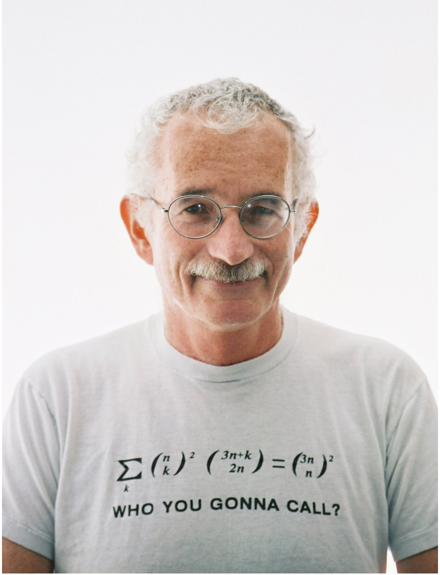
Joint work with Doron Zeilberger



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L. Euler (1707-1783)

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}.$$

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$$+ \frac{\frac{1}{18} - \frac{\sqrt{3}}{54}i}{x-e^{2\pi i/3}} + \frac{\frac{1}{18} + \frac{\sqrt{3}}{54}i}{x-e^{4\pi i/3}} + \frac{-\frac{1}{16}i}{x-i} + \frac{\frac{1}{16}i}{x+i}$$

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$$= \sum_{k=1}^{N} \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} \sum_{l=1}^{\lfloor N/k \rfloor} \frac{C_{h,k,l}(N)}{(x - e^{2\pi i h/k})^{l}}$$

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$$= \frac{C_{0,1,1}(4)}{x-1} + \frac{C_{0,1,2}(4)}{(x-1)^2} + \frac{C_{0,1,3}(4)}{(x-1)^3} + \frac{C_{0,1,4}(4)}{(x-1)^4} + \frac{C_{1,2,1}(4)}{x+1} + \frac{C_{1,2,2}(4)}{(x+1)^2} + \frac{C_{1,3,1}(4)}{x-e^{2\pi i/3}} + \frac{C_{2,3,1}(4)}{x-e^{4\pi i/3}} + \frac{C_{1,4,1}(4)}{x-i} + \frac{C_{3,4,1}(4)}{x+i}$$

Hans Rademacher (1892-1969)

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I propose the following conjecture."

Rademacher's Conjecture

For fixed relatively prime integers h and k with $0 \leq h < k$, and fixed positive integer l,

$$\lim_{N\to\infty} C_{h,k,l}(N)$$

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$$\Delta_\alpha^j = \sum_{h=0}^{j} (-1)^h \binom{j}{h} f(\alpha+j-h),$$

and $L_{3/2}(-y^2) = -\frac{1}{2\sqrt{\pi}y^2}\left(2\cos(2y) - \frac{\sin(2y)}{y}\right)$.

(Meager) evidence in favor

$$C_{0,1,1}(1) = -1$$

$$C_{0,1,1}(2) = -\frac{1}{4} = -0.25$$

$$C_{0,1,1}(3) = -\frac{17}{72} \approx -0.2361111111$$

$$C_{0,1,1}(4) = -\frac{17}{72} \approx -0.2361111111$$

$$C_{0,1,1}(5) = -\frac{20831}{86400} \approx -0.2410995370$$

$$\vdots$$

$$C_{0,1,1}(\infty) = -\frac{6}{25}\left(1 + \frac{2\sqrt{3}}{5\pi}\right) \approx -0.292927573960$$

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$$C_{0,1,1}(N+1) = \frac{-1}{(N+1)!} \times \sum_{h_1=1}^{1} \sum_{h_2=1}^{2} \cdots \sum_{h_N=1}^{N} \left(\rho_{i+1}^{-h_i}\right) H_N \left(\frac{\rho_2^{h_1}}{1-\rho_2^{h_1}}, \ldots, \frac{\rho_{N+1}^{h_N}}{1-\rho_{N+1}^{h_N}} \right),$$

where $\rho_j = e^{2\pi i/j}$, and $H_N(x_1, \ldots, x_n)$ is the Nth homogeneous symmetric function of $x_1, x_2, \ldots, x_n$.

Of the preceding formula, Andrews wrote,

"It may be reasonably objected that [it] . . . provides little hope of proving Rademacher's Conjecture. At most it may suggest the value of finding better formulas for $C_{h,k,l}(N)$."

“It is a capital mistake to theorise before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.”

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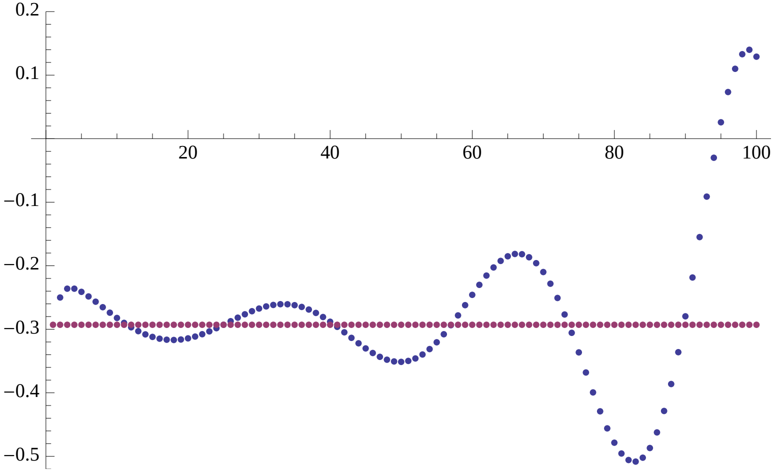
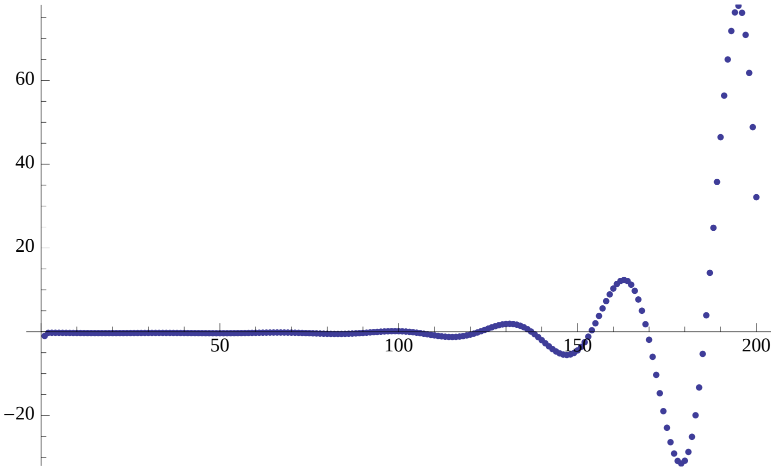
$C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 100$



$C_{0,1,1}(N)$ for $N = 1, 2, \ldots, 200$



S.-Z. revision of Rademacher's Conjecture

$C_{0,1,1}(N)$ is an oscillating function of N of "period" 32, with local maxima (resp. local minima) that attain arbitrary large positive (resp. negative) values as N increases.

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$$C_{0,1,1}(N) = \frac{e^{Nu}}{N^2}\left(\alpha \sin(\beta + Nv) + O(N^{-1})\right),$$

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The period of oscillations is $-2\pi/v \approx 31.9631$.

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For $N < 800$, the local maxima (resp. minima) occur when $N \equiv 12 - 9l \pmod{32}$ (resp. $N \equiv 28 - 9l \pmod{32}$).

Selected values of $C_{0,1,l}(N)$

$N\backslash l$	1	2	3	4	5
1	-1				
2	$-\frac{1}{4}$	$\frac{1}{2}$			
3	$-\frac{17}{72}$	$\frac{1}{4}$	$-\frac{1}{6}$		
4	$-\frac{17}{72}$	$\frac{59}{288}$	$-\frac{1}{8}$	$\frac{1}{24}$	
5	$-\frac{20831}{86400}$	$\frac{3}{16}$	$-\frac{31}{288}$	$\frac{1}{24}$	$-\frac{1}{120}$

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						$\frac{(-1)^{N+1}}{4(N-2)!}$	$\frac{(-1)^N}{N!}$

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				$\ddots$	$\ddots$	$\ddots$	
					$\frac{(-1)^N(9N^2-13N+26)}{288(N-2)!}$	$\frac{(-1)^{N+1}}{4(N-2)!}$	$\frac{(-1)^N}{N!}$

“Top down” conjecture for Rademacher’s coefficients

$$C_{0,1,N-r}(N) = \frac{(-1)^{N+r}}{4^r N! r!} P_{0,1,N-r}(N),$$

where, for $r > 0$, $P_{0,1,N-r}(N)$ is a convex, alternating, monic polynomial of degree $2r$ whose only real roots are 0 and 1.

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$$P_{0,1,N-3}(N) = N^6 - \frac{13}{3}N^5 + \frac{43}{3}N^4 - 25N^3 + \frac{98}{3}N^2 - \frac{56}{3}N$$

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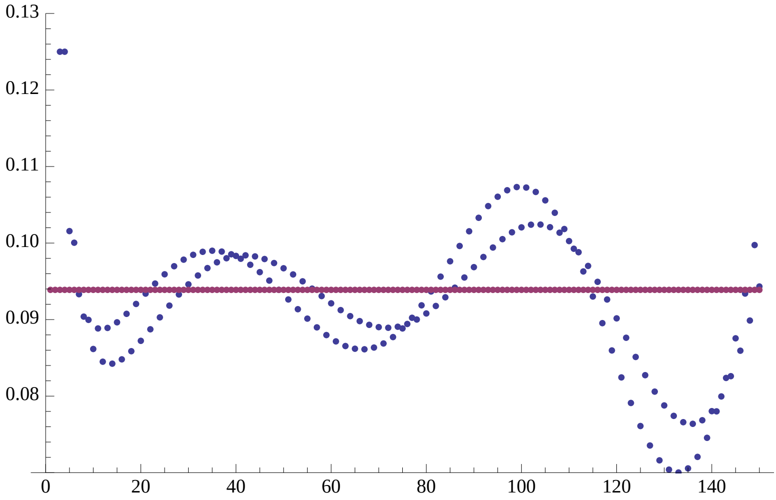
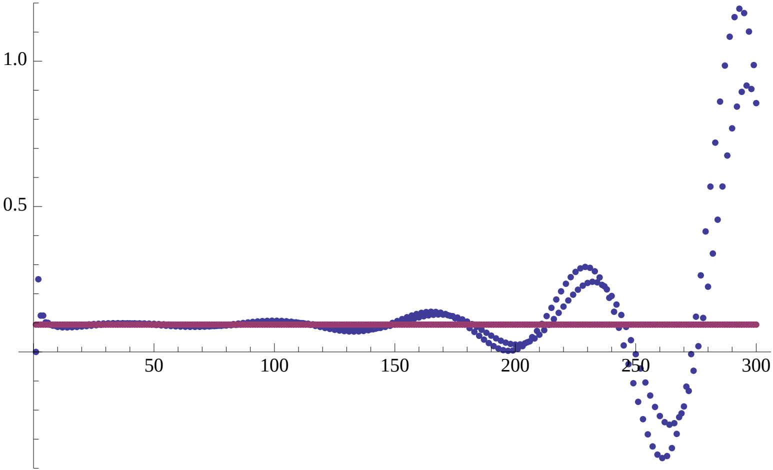
where B_j is the jth Bernoulli number, $\begin{bmatrix} r \\ k \end{bmatrix}$ is a Sterling number of the first kind, and

$$s_h(N) = \frac{1}{h+1} \sum_{j=0}^{h} \binom{h+1}{j+1} (-1)^{h-j} B_{h-j} N^{j+1}.$$

$C_{1,2,1}(N)$ for $N = 1, 2, \ldots, 150$



$C_{1,2,1}(N)$ for $N = 1, 2, \ldots, 300$



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Using the idea of "period 32," it appears that $C_{0,1,1}(32n)$ is strictly increasing. If we could show

$$\frac{C_{0,1,1}(32n+32)}{C_{0,1,1}(32n)} > 1,$$

for all sufficiently large n, then we'd be done.

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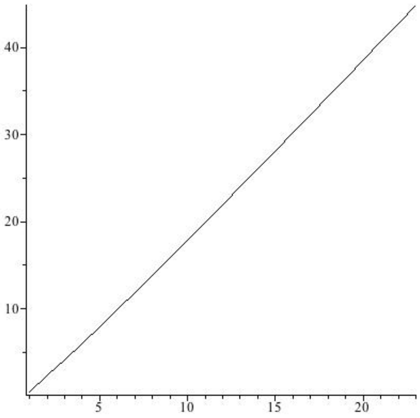
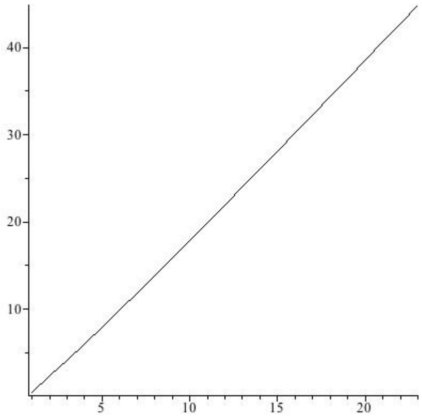


$\log C_{0,1,1}(32(n+3))$ as a function of n for $n = 1 \ldots 23$

How might we rigorously disprove RC?



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Linear regression: $\log C_{0,1,1}(32n) = 2.0280756n - 8.20939$

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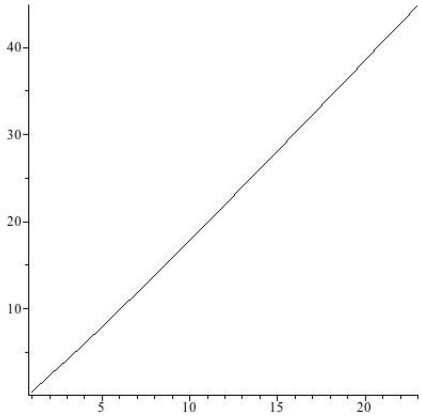

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Linear regression: $\log C_{0,1,1}(32n) = 2.0280756n - 8.20939$
Correlation coefficient: $r = 0.998549$.

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$$C_{0,1,1}(N) = \operatorname*{Res}_{x=1} \left(\sum_{n=0}^{\infty} p_N(n) x^n \right) = \operatorname*{Res}_{x=1} \frac{1}{(1-x)(1-x^2)\cdots(1-x^N)}$$

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$$\gamma_N : |x-1| = \frac{1}{N}.$$

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$$C_{0,1,1}(N) = \frac{1}{N}\int_0^1 \frac{e^{2\pi i t}\,dt}{\left(1-(1+\frac{e^{2\pi i t}}{N})\right)\ldots\left(1-(1+\frac{e^{2\pi i t}}{N})^N\right)}$$

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$$= 4(-1)^N N^{N-1}\int_{t=0}^{1/2} \frac{\cos(2\pi t(1-N))\,dt}{\Re\Gamma_{1+\exp(2\pi i t)/N}(N+1)}$$

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Then, estimate

$$\frac{C_{0,1,1}(32(n+1))}{C_{0,1,1}(32n)} = 32^{32}\left(\frac{n+1}{n}\right)^{32n-1}(n+1)^{32} \times \frac{\int_{t=0}^{1/2} \frac{\cos(2\pi(32n+31)t)dt}{\Re\Gamma_{1+\exp(2\pi it)/(32n+32)}(32n+33)}}{\int_{t=0}^{1/2} \frac{\cos(2\pi(32n-1)t)dt}{\Re\Gamma_{1+\exp(2\pi it)/(32n)}(32n+1)}}$$

Implications and applications of the Rademacher Coefficients

$$\sum_{n\geq 0} p_m(n) x^n = \sum_{k=1}^{m} \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \sum_{l=1}^{\lfloor m/k \rfloor} \frac{C_{h,k,l}(m)}{(x-\rho_{h,k})^l}$$

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Thus

$$p(n) = p_n(n) = \sum_{k=1}^{n} \sum_{\substack{0\leq h<k \\ \gcd(h,k)=1}} \sum_{l=1}^{\lfloor n/k \rfloor} (-1)^l \rho_{h,k}^{-l(n+1)} C_{h,k,l}(n) \binom{n+l-1}{n},$$

where $\rho_{h,k} := \exp(2\pi ih/k)$.