Integer Partitions and Identities of the Rogers-Ramanujan Type

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Assume throughout that $|q| < 1$.

Euler identities.

$$\sum_{n \geqq 0} \frac{q^{n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{j \geqq 1} (1+q^{2j-1})$$

$$\sum_{n \geqq 0} \frac{q^{n(n+1)}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{j \geqq 1} (1+q^{2j})$$

The Rogers-Ramanujan Identities.

$$\sum_{n \geqq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{j \geqq 1} \frac{1}{(1-q^{5j-1})(1-q^{5j-4})}$$

$$\sum_{n \geqq 0} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{j \geqq 1} \frac{1}{(1-q^{5j-2})(1-q^{5j-3})}$$

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

Thus

$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

$$(-q;q)_n = (1+q)(1+q^2)\cdots(1+q^n)$$

$$(q^2;q^2)_n = (1-q^2)(1-q^4)\cdots(1-q^{2n})$$

$$(q;q^2)_n = (1-q)(1-q^3)\cdots(1-q^{2n-1})$$

Euler identities.

$$\sum_{n\geqq 0} \frac{q^{n^2}}{(q^2;q^2)_n} = \prod_{j\geqq 1} \frac{(1-q^{4j-2})^2(1-q^{4j})}{1-q^j}$$

$$\sum_{n\geqq 0} \frac{q^{n(n+1)}}{(q^2;q^2)_n} = \prod_{j\geqq 1} \frac{(1-q^{4j-1})(1-q^{4j-3})(1-q^{4j})}{1-q^j}$$

The Rogers-Ramanujan Identities.

$$\sum_{n \geqq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{j \geqq 1} \frac{(1-q^{5j-2})(1-q^{5j-3})(1-q^{5j})}{1-q^j}$$

$$\sum_{n \geqq 0} \frac{q^{n(n+1)}}{(q;q)_n} = \prod_{j \geqq 1} \frac{(1-q^{5j-1})(1-q^{5j-4})(1-q^{5j})}{1-q^j}$$

Some Identities of Rogers-Ramanujan Type

$$\sum_{n \geqq 0} \frac{q^{n^2}}{(q;q^2)_n (q;q)_n} = \prod_{j \geqq 1} \frac{(1-q^{14j-6})(1-q^{14j-8})(1-q^{14j})}{1-q^j}$$

(Rogers, 1894)

$$\sum_{n \geqq 0} \frac{q^{n(3n-1)/2}}{(q;q^2)_n (q;q)_n} = \prod_{j \geqq 1} \frac{(1-q^{10j-4})(1-q^{10j-6})(1-q^{10j})}{1-q^j}$$

(Rogers, 1917)

The Jackson-Slater Identity:

$$\sum_{n \geqq 0} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{j \geqq 1} \frac{(1+q^{8j-3})(1+q^{8j-5})(1-q^{8j})}{1-q^{2j}}$$

(Jackson, 1928)

$$\sum_{n \geqq 0} \frac{q^{n(n+1)}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_n} = \prod_{j \geqq 1} \frac{(1-q^{6j})}{(1-q^j)(1-q^{6j-3})}$$

(Bailey, 1936)

$$\sum_{n \geqq 0} \frac{q^{n(n+1)}(q^3;q^3)_n}{(q;q)_{2n+1}(q;q)_n} = \prod_{j \geqq 1} \frac{1-q^{9j}}{1-q^j}$$

(Dyson, 1947)

$$\sum_{n \geqq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \prod_{j \geqq 1} \frac{1}{(1-q^{8j-1})(1-q^{8j-4})(1-q^{8j-7})}$$

(Slater, 1952)

$$\sum_{n\geqq 0} \frac{q^{n(n+2)}(q^3;q^6)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}(q;q^2)_n}$$

$$= \prod_{j\geqq 1} \frac{(1-q^{12j-2})(1-q^{12j-6})(1-q^{12j-10})(1-q^{12j})}{1-q^j}$$

(Slater, 1952)

A pair of sequences $(\alpha_n(x,q), \beta_n(x,q))$ is called a *Bailey pair* if

$$\beta_n(x,q) = \sum_{r=0}^{n} \frac{\alpha_r(x,q)}{(q;q)_{n-r}(xq;q)_{n+r}}.$$

Bailey's Lemma–Weak Form. *If* $(\alpha_n(x,q), \beta_n(x,q))$ *form a Bailey pair, then*

$$\sum_{n \geqq 0} x^n q^{n^2} \beta_n(x,q) = \frac{1}{\prod_{j \geqq 1}(1 - xq^j)} \sum_{r \geqq 0} x^r q^{r^2} \alpha_r(x,q).$$

Strategy: Find Bailey pairs such that the β_n is "nice" and the corresponding corresponding α_n, when inserted into the RHS of the weak Bailey lemma sums via . . .

Jacobi's Triple Product Identity. *If* $z \neq 0$ *and* $|y| < 1$, *then*

$$\sum_{r=-\infty}^{\infty} (-1)^r z^r y^{r^2} = \prod_{j \geqq 1} (1 - zy^{2j-1})(1 - z^{-1}y^{2j-1})(1 - y^{2j}).$$

The standard multiparameter Bailey pair (SMBP):

$$\alpha_n^{(d,e,k)}(x,q) := \frac{(-1)^{n/d} x^{(k/d-1)n/e} q^{(k/d-1+1/2d)n^2/e-n/2e}}{(1-x^{1/e})(q^{d/e};q^{d/e})_{n/d}} \times (1-x^{1/e}q^{2n/e})(x^{1/e};q^{d/e})_{n/d}\ \chi(d \mid n),$$

where

$$\chi(P(n,d)) = \begin{cases} 1 & \text{if } P(n,d) \text{ is true,} \\ 0 & \text{if } P(n,d) \text{ is false} \end{cases}$$

(d, e, k)	Author/Identity	modulus
$(1, 1, 2)$	Rogers-Ramanujan	5
$(1, 1, k)$	G.E. Andrews	$2k + 1$
$(1, 2, 1)$	L.J. Rogers	5
$(1, 2, 2)$	Rogers-Selberg	7
$(1, 2, 4)$	D. Stanton	11
$(1, 3, 2)$	W.N. Bailey	9
$(1, 6, 3)$	Verma-Jain	17
$(1, 6, 4)$	Verma-Jain	19
$(2, 1, 2)$	L.J. Rogers	10
$(2, 1, 3)$	L.J. Rogers	14
$(2, 1, 5)$	Verma-Jain	22
$(2, 2, 3)$	S. O. Warnaar	11
$(2, 2, 4)$	G. E. Andrews	13
$(2, 3, 4)$	Verma-Jain	34
$(2, 3, 5)$	Verma-Jain	38
$(3, 1, 4)$	F.J. Dyson	27
$(3, 1, 5)$	Verma-Jain	33
$(3, 2, 5)$	Verma-Jain	51
$(3, 2, 6)$	Verma-Jain	57
$(6, 1, 8)$	Verma-Jain	102
$(6, 1, 9)$	Verma-Jain	114

(d, e, k)	modulus
$(1, 2, 3)$	9
$(1, 3, 1)$	7
$(1, 3, 3)$	11
$(1, 3, 5)$	15
$(1, 4, 1)$	9
$(1, 4, 2)$	11
$(1, 4, 4)$	15
$(2, 1, 1)$	6
$(2, 1, 4)$	18
$(2, 2, 2)$	18
$(2, 2, 5)$	30
$(3, 1, 3)$	21
$(4, 1, 6)$	52
$(4, 1, 7)$	60

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+3r^2+4nr}}{(-q;q)_{n+r}(-q;q)_{n+2r}(q;q)_n(q;q)_r} = \prod_{j\geqq 1} \frac{(1-q^{9j-4})(1-q^{9j-5})(1-q^{9j})}{1-q^{2j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+3r^2+4nr}(-q;-q)_{2n+2r}}{(q^2;q^2)_{2n+2r}(q^2;q^2)_r(q^2;q^2)_n} = \prod_{j\geqq 1} \frac{(1-q^{14j-6})(1-q^{14j-8})(1-q^{14j})(1-q^{4j-2})}{1-q^j}$$

$$\sum_{n,r\geqq 0}\frac{q^{2n^2+3r^2+4nr}}{(-q;q)_{2n+2r}(q;q^2)_r(q;q)_r(q^2;q^2)_n}$$
$$=\prod_{j\geqq 1}\frac{(1-q^{30j-14})(1-q^{30j-16})(1-q^{30j})}{1-q^{2j}}$$

$$\sum_{n,r\geqq 0}\frac{q^{n^2+2r^2}}{(q;q^2)_n(q^2;q^2)_r(q^2,q^4)_r(q;q)_{n-2r}}$$
$$=\prod_{j\geqq 1}\frac{(1-q^{60j-28})(1-q^{60j-32})(1-q^{60j})}{1-q^j}$$

Definition 1. A *partition* π of an integer n is a finite, nonincreasing sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_s)$ whose sum is n. Each π_j is called a *part* of π.

Example 1. There are five partitions of the number 4:

$$(4), \quad (3,1) \quad (2,2) \quad (2,1,1) \quad (1,1,1,1)$$

Recall the first Rogers-Ramanujan identity:

$$\sum_{n \geqq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{j \geqq 1} \frac{1}{(1-q^{5j-1})(1-q^{5j-4})}.$$

The First Rogers-Ramanujan Identity—-Combina Form.

The number of partitions of n into parts which are distinct, nonconsecutive integers
equals
the number of partitions of n into parts which are $\equiv \pm 1 \pmod 5$.

Example: Of the forty-two partitions of 10, there are six in which the parts are distinct and nonconsecutive:

$$(10), (9, 1), (8, 2), (7, 3), (6, 4), (6, 3, 1)$$

and there are six in which all parts are $\equiv \pm 1$ (mod 5):

$$(9, 1), (6, 4), (6, 1, 1, 1, 1), (4, 4, 1, 1),$$

$$(4, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1).$$

Gordon's Combinatorial Generalization of Rogers-Ramanujan. *Let* $B_{k,i}(n)$ *denote the number of partitions of* n *wherein*

- *1 appears as a part at most* $i-1$ *times, and*
- *the total number of appearances of any two consecutive integers* j *and* $j+1$ *is at most* $k-1$

Let $A_{k,i}(n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm i \pmod{2k+1}$*. Then*

$$A_{k,i}(n) = B_{k,i}(n)$$

for all $n \in \mathbb{Z}$ *and* $1 \leqq i \leqq k$*.*

Andrews-Gordon Theorem. *For* $1 \leqq i \leqq k,\ k \geqq 1$,

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geqq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_{k-1}}} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm i \ (\mathrm{mod}\ 2k+1)}} \frac{1}{1 - q^j}$$

where $N_h = n_h + n_{h+1} + \cdots + n_{k-1}$.

$$H_{k,i}(a;x;q) := \left(\prod_{j \geqq 1} \frac{1-axq^{j}}{1-xq^{j-1}}\right) \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+(1-i)n} a^{n} (a^{-1};q)_n}{(q;q)_n (axq;q)_n} \times (1 - x^{i} q^{2ni})(x;q)_n$$

$$J_{k,i}(a;x;q) := H_{k,i}(a;xq;q) - axqH_{k,i-1}(a;xq;q)$$

$$\sum_{n=0}^{\infty} x^{en} q^{en^2} \beta_n^{(d,e,k)}(x^e, q^e) = \left(\prod_{j \geqq 1} \frac{1 - xq^{dj}}{1 - x^e q^{ej}} \right) H_{d(e-1)+k,1}(0; x; q^d)$$

$$Q_i^{(d,e,k)}(x) := \left(\prod_{j \geqq 1} \frac{1 - xq^{dj}}{1 - x^e q^{ej}} \right) J_{d(e-1)+k,i}(0; x; q^d)$$

Let $A_{d,k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm di \pmod{2dk+d}$. Let $B_{d,k,i}(n)$ denote the number of partitions of n wherein

- The integer d appears as a part at most $i-1$ times,
- the total number of appearances of dj and $dj+d$ (i.e. any two consecutive multiples of d) together is at most $k-1$, and
- nonmultiples of d may appear as parts without restriction.

Then for $1 \leqq i \leqq k$, $A_{d,k,i}(n) = B_{d,k,i}(n)$

Euler multiparameter Bailey pair

$$\widetilde{\alpha}_n^{(d,e,k)}(x,q) := q^{n(d-n)/2de} x^{-n/de} \frac{(-x^{1/e}; q^{d/e})_{n/d}}{(-q^{d/e}; q^{d/e})_{n/d}} \, \alpha_n^{(d,e,k)}(x,q)$$

Jackson-Slater multiparameter Bailey pair

$$\bar{\alpha}_n^{(d,e,k)}(x,q) := (-1)^{n/d} q^{-n^2/2de} \frac{(q^{d/2e}; q^{d/e})_{n/d}}{(x^{1/e} q^{d/2e}; q^{d/e})_{n/d}} \, \alpha_n^{(d,e,k)}(x,q)$$

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+4nr+4r^2}(q;q^2)_r}{(q^2;q^2)_r(-q;q)_{2r}(q^2;q^2)_n} = \prod_{j\geqq 1} \frac{(1-q^{5j})(1-q^{10j-5})}{1-q^{2j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+2r^2}(-q;q^2)_r}{(q;q)_{2r}(q;q)_n} = \prod_{j\geqq 1} \frac{(1-q^{10j})(1-q^{20j-10})}{1-q^{j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2nr+3r^2}(-q;q^2)_{n+r}(-q^2;q^4)_r}{(q^2;q^2)_{2r}(q^2;q^2)_n} = \prod_{j\geqq 1} \frac{(1-q^{14j})(1-q^{28j-14})}{1-q^{2j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{n^2+2r^2}}{(q;q^2)_n(q;q)_{2r}(q;q)_{n-2r}} = \prod_{j\geqq 1} \frac{(1+q^{16j-7})(1+q^{16j-9})(1-q^{16j})}{1-q^j}$$

$$\sum_{n,r\geqq 0} \frac{q^{4n^2+6r^2+8nr}}{(-q^2;q^2)_{2n+2r}(q;q)_{2r}(q^4;q^4)_n} = \prod_{j\geqq 1} \frac{(1+q^{16j-7})(1+q^{16j-9})(1-q^{16j})}{1-q^{4j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{4n^2+8r^2+8nr}(-q;q^2)_{2r}}{(q^4;q^4)_{2r}(q^4;q^4)_n} = \prod_{j\geqq 1} \frac{(1+q^{20j-9})(1+q^{20j-11})(1-q^{20j})}{1-q^{4j}}$$

$$\sum_{n,r\geqq 0} \frac{q^{2n^2+4r^2+4nr}}{(-q;q)_{2n+2r}(q;q)_{2r}(q;q)_{2n}} = \prod_{j\geqq 1} \frac{(1+q^{24j-11})(1+q^{24j-13})(1-q^{24j})}{1-q^{2j}}$$

Andrews-Bressoud Even Modulus Partition Thm. *Let* $1 \leqq i < k$. *Let* $C_{k,i}(n)$ *denote the number of partitions* $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ *of* n *such that*

- 1 *appears at most* $i-1$ *times,*

- $\pi_j - \pi_{j+k-1} \geqq 2$, *and*

- *if* $\pi_j - \pi_{j+k-2} \leqq 1$, *then* $\sum_{h=0}^{k-2} \pi_{j+h} \equiv (i-1) \pmod{2}$.

Let $D_{k,i}(n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm i \pmod{2k}$. *Then* $C_{k,i}(n) = D_{k,i}(n)$ *for all* n.

Andrews-Santos Combinatorial Generalization of Jackson-Slater. *Let* $1 \leqq i \leqq k$. *Let* $E_{k,i}(n)$ *denote the number of partitions of* n *such that*

- 2 *appears as a part at most* $i-1$ *times,*

- *the total number of appearances of any two consecutive evens is at most* $k-1$,

- *for any positive integer* j, $2j-1$ *may appear only if* $2j-2$ *and* $2j$ *appears the maximum number of times, i.e.* $k-1$ *times.*

Let $G_{k,i}(n)$ *denote the number of partitions of* n *into parts which are either even but not multiples of* $4k$, *or distinct, odd, and congruent to* $\pm(2i-1) \pmod{4k}$. *Then* $E_{k,i}(n) = G_{k,i}(n)$ *for all* n.

Definition 2. An *overpartition* of an integer n is a nonincreasing finite sequence of positive integers whose sum is n in which the first occurence of a given number may be overlined

Example 2. There are three ordinary partitions of 3:

$$(3) \quad (2,1) \quad (1,1,1)$$

and there are eight overpartitions of 3:

$$(3) \quad (\bar{3})$$
$$(2,1) \quad (\bar{2},1) \quad (2,\bar{1}) \quad (\bar{2},\bar{1})$$
$$(1,1,1) \quad (\bar{1},1,1)$$

Let d be an odd positive integer and let $1 \leqq i \leqq k$. Let $E_{d,k,i}(n)$ denote the number of partitions π of n such that

- The number $2d$ appears as a part at most $i-1$ times,
- The total number of consecutive multiples of $2d$ is at most $k-1$, and
- An odd multiple of d may appear only if the total number of adjacent even multiples of d appears exactly $k-1$ times

Let $G_{d,k,i}(n)$ denote the number of partitions of n into parts which are either even but not multiples of $4dk$, or distinct, odd and congruent to $\pm d(2i-1) \pmod{4dk}$.
Then $E_{d,k,i}(n) = G_{d,k,i}(n)$.

Let d be an even positive integer and let $1 \leqq i \leqq k$. Let $I_{d,k,i}(n)$ denote the number of partitions π of n such that

- The number of appearances of any odd number must be a multiple of d
- Multiples of d appear at most $i-1$ times
- The total number of consecutive multiples of $2d$ is at most $k-1$, and
- An odd multiple of d may appear only if the total number of adjacent even multiples of d appears exactly $k-1$ times

Let $K_{d,k,i}(n)$ denote the number of overpartitions of n into parts which are either even but not multiples of $4dk$, and the overlined parts are congruent to $\pm d(2i-1) \pmod{4dk}$.
Then $I_{d,k,i}(n) = K_{d,k,i}(n)$.