A Partition Bijection Related to the Rogers-Selberg Identities

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Québec-Maine Number Theory Conference
October 1, 2006

The Rogers-Ramanujan Identities.

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 1 \pmod 5}} \frac{1}{1-q^j}$$

and

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 2 \pmod 5}} \frac{1}{1-q^j}.$$

Assume throughout that $|q| < 1$.

Rising q-factorial notation

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$$

The Rogers-Ramanujan Identities.

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 1 \pmod 5}} \frac{1}{1-q^j}$$

and

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 2 \pmod 5}} \frac{1}{1-q^j}.$$

The Rogers-Selberg Identities.

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+2};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 1 \pmod 7}} \frac{1}{1-q^j},$$

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}(-q^{2j+1};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 2 \pmod 7}} \frac{1}{1-q^j},$$

and

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(-q^{2j+1};q)_\infty}{(q^2;q^2)_j} = \prod_{\substack{j \geqq 1 \\ j \not\equiv 0, \pm 3 \pmod 7}} \frac{1}{1-q^j}.$$

A *partition* π of an integer n is a nonincreasing finite sequence of positive integers

$$\pi = \{\pi_1, \pi_2, \pi_3, \ldots, \pi_s\}$$

such that $\sum_i^s \pi_i = n$.

Each nonzero term in $\{\pi_1, \pi_2, \pi_3, \ldots, \pi_s\}$ is called a *part* of the partition π.

The seven partitions of 5 are thus

$$\{5\} \qquad \{4,1\} \qquad \{3,2\} \qquad \{3,1,1\}$$

$$\{2,2,1\} \qquad \{2,1,1,1\} \qquad \{1,1,1,1,1\}.$$

The *multiplicity* of the integer j in the partition π, denoted $m_j(\pi)$, is the number of times j appears in π.

$$\pi = \langle 1^{m_1(\pi)} 2^{m_2(\pi)} 3^{m_3(\pi)} \dots \rangle$$

The seven partition of 5 are thus

$$\langle 5 \rangle \qquad \langle 1\ 4 \rangle \qquad \langle 2\ 3 \rangle \qquad \langle 1^2 3 \rangle$$

$$\langle 1\ 2^2 \rangle \qquad \langle 1^3 2 \rangle \qquad \langle 1^5 \rangle.$$

The Rogers-Ramanujan Identities—Combinatorial Version. *For* $i = 1, 2$*,*

the number of partitions of n *into parts which are nonconsecutive integers greater than* $2 - i$ *and in which no part is repeated*

equals

the number of partitions of n *into parts* $\not\equiv 0, \pm i \pmod 5$*.*

Gordon's Theorem.
Let $G_{k,i}(n)$ *denote the number of partitions of* n *into parts such that* 1 *appears as a part at most* $i-1$ *times and the total number of appearances of any two consecutive integers is at most* $k-1$.

Let $C_{k,i}(n)$ *denote the number of partitions of* n *into parts* $\not\equiv 0, \pm i \pmod{2k+1}$.

Then $G_{k,i}(n) = C_{k,i}(n)$ *for* $1 \leqq i \leqq k$ *and all integers* n.

Theorem 1 (Andrews).
Let $A_2(n)$ denote the number of partitions of n such that if $2j$ is the largest repeated even part, then all positive even integers less than $2j$ also appear at least twice, no odd part less than $2j$ appears, and no part greater than $2j$ is repeated.

Then $A_2(n) = C_{3,2}(n)$ for all n.

Let $\mathcal{G}_2$ denote the set of partitions enumerated by $G_{3,2}(n)$ in Gordon's theorem, i.e. partitions π such that

$$m_1(\pi) \leqq 1$$

and

$$m_j(\pi) + m_{j+1}(\pi) \leqq 2$$

for all $j \geqq 1$.

Let $\mathcal{A}_2$ denote the set of partitions enumerated by $A_2(n)$ in Theorem 1, i.e. partitions π such that

$$m_j(\pi) \leqq 1 \text{ if } j \text{ is odd,}$$

$$m_j(\pi) = 0 \text{ if } j \text{ is odd and } j < R(\pi), \text{ and}$$

$$m_j(\pi) \geqq 2 \text{ if } j \text{ is even and } j < R(\pi),$$

where $R(\pi)$ is the largest repeated part in π.

A partition $\pi \in \mathcal{G}_2$ is one in which

- no number appears more than twice as a part,

- if r appears twice, then neither $r-1$ nor $r+1$ appear, and

- 1 appears at most once.

A partition $\pi \in \mathcal{A}_2$ may be thought of as a union of two partitions :

- a partition into 2's, 4's, 6's, . . . , $2j$'s with all parts repeated, and

- a partition into distinct parts greater than $2j$.

Example.

$$\{30, 27, 19, 15, 15, 13, 12, 10, 8, 8, 4, 4, 2, 1\} \in \mathcal{G}_2$$

$$\downarrow$$

$$\{30, 27, 19, 15, 14, 12, 8, 7\} \cup \langle 2^4 4^4 6^2 \rangle \in \mathcal{A}_2$$

30	27	19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
		+	13	12	10	2	2	2	2	2	2	2	2				
					+	2	2	2	2								
					+	2	1										
30	27	19	15	14	12	8	7	6	6	4	4	4	4	2	2	2	2

18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
30	27	19	15	14	12	8	7	6	6	4	4	4	4	2	2	2	2
			$\uparrow$														

So first row is

30 27 19 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

11	10	9	8	7	6	5	4	3	2	1
13	12	10	6	5	4	4	2	2	2	2
			↑							

So first row and second rows are

30	27	19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
			13	12	10	2	2	2	2	2	2	2	2				

$$
\begin{array}{cccc}
4 & 3 & 2 & 1 \\
\hline
4 & 3 & 2 & 2 \\
\uparrow & & &
\end{array}
$$

So the first three rows are

30	27	19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
			13	12	10	2	2	2	2	2	2	2	2				
						2	2	2	2								

2	1
2	1
↑	

30	27	19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
		+	13	12	10	2	2	2	2	2	2	2	2				
					+	2	2	2	2								
					+	2	1										
30	27	19	15	14	12	8	7	6	6	4	4	4	4	2	2	2	2

Analogous interpretations of the other two Rogers-Selberg identities can be given and they in turn can be mapped similarly to the $i = 1$ and $i = 3$ instances of the partitions enumerated by the $k = 3$ case of Gordon's theorem.