Parts and Subword Patterns in Integer Compositions

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Co-authors

Joint work with

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Definitions

A *composition* σ of a positive integer n is a tuple

$$(\sigma_1, \sigma_2, \dots, \sigma_l)$$

of positive integers such that

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$$\mathscr{C}_4 = \{(4), (3,1), (1,3), (2,2), (2,1,1), (1,2,1), (1,1,2), (1,1,1,1)\}.$$

A *partition* of n is a composition

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_l)$$

of n in which $\sigma_i \leq \sigma_{i+1}$ for $i = 1, 2, \dots, l-1$.

Odds and "Runs"

14 odds:

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Counting odds and "runs"

n	$OP(n) = R(n)$	$EP(n)$
1	1	0
2	2	1
3	6	2
4	14	6
5	34	14
6	78	34
7	178	78
8	398	178
9	882	398
10	1934	882

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Evens and levels

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Bijection between even parts and levels

even part $2k$	the level k, k
$\underline{4}$	$\underline{22}$
$\underline{2}2$	$\underline{11}2$
$2\underline{2}$	$2\underline{11}$
$\underline{2}11$	$\underline{11}11$
$1\underline{2}1$	$1\underline{11}1$
$11\underline{2}$	$11\underline{11}$

The number of parts that are multiples of m in $\mathscr{C}_n$ equals the number of instances of m consecutive equal parts in $\mathscr{C}_n$.

Is there a nice bijection between odd parts and runs?

$$|\mathscr{C}_{n+1}| = 2|\mathscr{C}_n|.$$

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$$\{3, 21, 12, 111\}, \{3, 21, 12, 111\}$$
$$\downarrow$$
$$\{13, 121, 112, 1111, 4, 31, 22, 211\}$$

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- $IL(n) :=$ number of initial levels, i.e. number of compositions σ of n such that $\sigma_1 = \sigma_2$
- $IS(n) :=$ number of initial "steps," i.e. number of compositions σ of n such that $\sigma_1 + 1 = \sigma_2$

Induction Proof

$$OP(n+1) = (OP(n) + |\mathscr{C}_n|) + (OP(n) + IE(n) - IO(n))$$

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$$\begin{aligned}(OP(n) + |\mathscr{C}_n|) + (OP(n) + IE(n) - IO(n))\\ = (R(n) + IB(n)) + (R(n) + IL(n) - IS(n))\end{aligned}$$

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iff (by I.H. and $|\mathscr{C}_n| = I1(n) + IB(n)$)

$$IE(n) - IO(n) = IL(n) - IS(n) - I1(n)$$

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iff (by I.H. and $|\mathscr{C}_n| = \mathbb{1}(n) + \mathbb{B}(n)$)

$$\mathbb{E}(n) - \mathbb{O}(n) = \mathbb{L}(n) - \mathbb{S}(n) - \mathbb{1}(n)$$

iff

$$\cancel{\mathbb{E}(n)} - \mathbb{O}(n) = \cancel{\mathbb{L}(n)} - \mathbb{S}(n) - \mathbb{1}(n)$$

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$$(k, \dots) \leftrightarrow \left(\frac{k-1}{2}, \frac{k+1}{2}, \dots\right)$$

The mapping for $n = 4$

odd part	run
1111	4
1111	13
1111	112
1111	1111
112	31
112	112
121	121
13	13
13	121
211	22
211	211
31	211
31	31

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- Let $P(i, m; n)$ denote the number of parts congruent to $i \pmod{m}$ in $\mathscr{C}_n$, where $1 \leq i \leq m$.
- Let $S(\tau; n)$ denote the number of subword pattern matches to τ in $\mathscr{C}_n$.
- Unlike with the usual pattern matching/avoidance, the patterns must occur in consecutive parts.
- For example, a *level* is the subword matching the pattern $\tau = 11$.

Let $\tau = \tau_1 \tau_2 \cdots \tau_l$ be a subword pattern of length l in which

$$1 = \tau_1 \le \tau_2 \le \cdots \le \tau_l \le l.$$

and $S(\tau; n)$ is as before. If we simultaneously think of τ as a partition and $|\tau|$ is its weight, then

$$\sum_{n \ge 0} S(\tau; n) x^n = \frac{x^{|\tau|}(1-x)^2}{(1-2x)^2(1-x^l)} \prod_{j=1}^{l-1} \frac{1}{(1-x^{l-j})^{\tau_{j+1}-\tau_j}}.$$

Proof

Observe that

$$\frac{x^{|\tau|}(1-x)^2}{(1-2x)^2(1-x^l)}\prod_{j=1}^{l-1}\frac{1}{(1-x^{l-j})^{\tau_{j+1}-\tau_j}}$$

$$= \frac{1-x}{1-2x}\cdot\frac{x^l}{1-x^l}\prod_{j=1}^{l-1}\frac{x^{(l-j)(\tau_{j+1}-\tau_j)}}{(1-x^{l-j})^{\tau_{j+1}-\tau_j}}\cdot\frac{1-x}{1-2x}. \quad (1)$$

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Note that the first and last factors of (1) are both

$$\frac{1-x}{1-2x} = \sum_{n\geq 0} |\mathscr{C}_n| x^n$$

Proof

while the middle factor of (1) is

$$\frac{x^l}{1-x^l}\prod_{j=1}^{l-1}\frac{x^{(l-j)(\tau_{j+1}-\tau_j)}}{(1-x^{l-j})^{\tau_{j+1}-\tau_j}} = \sum_{n\geq 0}|\mathscr{P}_n(T)|x^n$$

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where T is the set of parts appearing in τ', when τ is interpreted as a partition rather than a subword pattern and $\mathscr{P}_n(T)$ is the set of all partitions of n with parts in T and each member of T appears at least once as a part.

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Example

Table : An example of the correspondence for $S(112223, 13) = 8$.

member of $\mathscr{C}_{13}$	corresponding ordered triple
11112223	(11, 146, ∅)
11122231	(1, 146, 1)
1112224	(1, 1146, ∅)
11222311	(∅, 146, 11)
1122232	(∅, 146, 2)
1122241	(∅, 1146, 1)
112225	(∅, 11146, ∅)
2112223	(2, 146, ∅)

Generating Functions

$$\sum_{n \geq 1} P(i,m;n)x^n = \frac{x^i(1-x)^2}{(1-2x)^2(1-x^m)}$$

For all $n \in \mathbb{N}$,

$$P(1,2;n) = S(1;n) - S(11;n)$$
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$$P(1,3;n) = S(1;n) - S(11;n) - S(1^3;n) + S(112;n) - 2S(122;n)$$
$$P(2,3;n) = S(11;n) - S(112;n) + 2S(122;n)$$
$$P(3,3;n) = S(1^3;n)$$

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$$P(2,3;n) = S(11;n) - S(112;n) + 2S(122;n)$$
$$P(3,3;n) = S(1^3;n)$$

$$P(1,4;n) = S(1;n) - S(11;n) - S(1^3;n) + S(1122;n) - S(12^3;n) - S(1223;n) + 2S(1233;n)$$
$$P(2,4;n) = S(11;n) - S(1^4;n)$$
$$P(3,4;n) = S(1^3;n) - S(1122;n) + S(12^3;n) + S(1223;n) - 2S(1233;n)$$
$$P(4,4;n) = S(1^4;n)$$

Enumeration

$$P(i, m; n) = \left\lfloor \frac{2^{n+m-i-2}\Big((n-i+3)(2^m-1)-m\Big)}{(2^m-1)^2} \right\rceil,$$

where $\lfloor x \rceil$ is the integer nearest to x.

Thank you for your attention!