Euler type partition recurrences

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Joint work with Yuriy Choliy

A *partition* λ of an integer n is a finite, weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers (called the *parts* of λ) that sum to n.

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Thus the seven partitions of 5 are

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Let $p(n)$ denote the number of partitions of the integer n.

Euler proved the following recurrence for $p(n)$ [**?**, p. 12, Cor. 1.8]:

$$p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)$$
$$+\cdots+(-1)^j p(n-j(3j-1)/2)+(-1)^j p(n-j(3j+1)/2)+\cdots$$
$$=\begin{cases} 1 & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

Note that here and throughout, x represents a formal variable.

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The generating function for $p(n)$

Let $P(x) := \sum_{n\geq 0} p(n)x^n$. *Then*

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The pentagonal numbers theorem

$$\frac{1}{P(x)} = \sum_{k\in\mathbb{Z}} (-1)^k x^{k(3k-1)/2}.$$

$$1 = P(x) \cdot \frac{1}{P(x)} = \left(\sum_{j \geq 0} p(j) x^j \right) \left(\sum_{k \in \mathbb{Z}} (-1)^k x^{k(3k-1)/2} \right)$$

$$\sum_{n \geq 0} \left(\sum_{j \in \mathbb{Z}} (-1)^j p\Big(n - j(3j-1)/2\Big) \right) x^n, \quad (2)$$

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$$\sum_{n \geq 0} \left(\sum_{j \in \mathbb{Z}} (-1)^j p\Big(n - j(3j-1)/2 \Big) \right) x^n, \quad (2)$$

and (1) follows from extracting the coefficient of x^n from the extremes of (2).

Theorem

For all integers n,

$$p(n) - p(n-1) - p(n-3) + p(n-6) + p(n-10) - p(n-15)$$
$$- p(n-21) + \cdots + (-1)^j p(n-(2j^2-j)) + (-1)^j p(n-(2j^2+j)) + \cdots$$
$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ q(n/2) & \text{if } n \text{ is even} \end{cases},$$

where $q(n)$ denotes the number of partitions of n into distinct parts.

Theorem

For all integers n,

$$p(n)-p(n-1)-p(n-2)+p(n-4)+p(n-8)-p(n-9)-p(n-18) + \cdots + (-1)^j p(n-j^2) + (-1)^j p(n-2j^2) + \cdots = \begin{cases} 0 & \text{if } n \text{ is odd} \\ qq(n) & \text{if } n \text{ is even} \end{cases}, \quad (3)$$

where $qq(n)$ denotes the number of partitions of n into distinct, odd parts.

Two-color partitions

The five two-color partitions of 2 are

$$(\color{red}{2}) \quad (\color{blue}{2}) \quad (\color{red}{1}, \color{red}{1}) \quad (\color{red}{1}, \color{blue}{1}) \quad (\color{blue}{1}, \color{blue}{1}).$$

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Let $p_2(n)$ denote the number of two-color partitions of n.

Theorem

If n is even, then

$$p(n) = \sum_{i \geq 1} p_2 \left(\frac{n - t_i^e}{2} \right),$$

where t_i^e is the ith even triangular number ($t_1^e = 0$, $t_2^e = 6$, $t_3^e = 10$, etc.), i.e.

$$t_i^e = \frac{(2i-1)(2i-1+(-1)^i)}{2}.$$

Theorem

If n is odd, then

$$p(n) = \sum_{i \geq 1} p_2\left(\frac{n - t_i^o}{2}\right),$$

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An *overpartition* of n is a two-color partition of n where a given positive integer in the second color may appear at most once as a part. (Parts in the first color may be repeated any number of times.)

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The eight overpartitions of 3 are as follows:

$$(3) \quad (\overline{3}) \quad (2,1) \quad (\overline{2},1) \quad (2,\overline{1}) \quad (\overline{2},\overline{1}) \quad (1,1,1) \quad (1,1,\overline{1}).$$

Theorem

Let $\overline{p}(n)$ denote the number of overpartitions of n. Let $v(n)$ denote the sequence determined by pairing the left side of (7) *with the right side of* (1)*, namely*

$$v(n)-v(n-1)-v(n-2)+v(n-4)+v(n-8)-v(n-9)-v(n-18) + \cdots + (-1)^j v(n-j^2) + (-1)^j v(n-2j^2) + \cdots = \begin{cases} 1 & \text{if } n=0, \text{ and} \\ 0 & \text{otherwise} \end{cases} . \quad (4)$$

Then $v(n) = \overline{p}(n/2)$ if n is even.

$$p(n) \equiv \sum p\left(\frac{n-t}{4}\right) \pmod 2, \tag{5}$$

where the sum is taken over all t such that t is triangular, $0 \le t \le n$, and $t \equiv n \pmod 4$.

MacMahon's parity of $p(n)$ recurrence

$$p(n) \equiv \sum p\left(\frac{n-t}{4}\right) \pmod 2, \qquad (5)$$

where the sum is taken over all t such that t is triangular, $0 \le t \le n$, and $t \equiv n \pmod 4$.

MacMahon's recurrence (5) follows easily from the observation that $P(x) \equiv P(x^4)\psi(x) \pmod 2$, where

$$\psi(x) := 1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \cdots .$$

Theorem

$$\sum_{j\geq 0} p(n-4\pi_j) \equiv \begin{cases} 1 \pmod 2 & \textit{if } n \textit{ is a triangular number,} \\ 0 \pmod 2 & \textit{otherwise} \end{cases} \tag{6}$$

where π_j *is the* j*th generalized pentagonal number, i.e.*

$$\pi_j = \frac{3}{8}j^2 + \frac{3-(-1)^j}{8}j + \frac{1-(-1)^j}{16}.$$

Triangular Number Recurrence

Theorem

For all integers n,

$$p(n)-p(n-1)-p(n-3)+p(n-6)+p(n-10)-p(n-15)-p(n-21) + \cdots + (-1)^j p(n-(2j^2-j)) + (-1)^j p(n-(2j^2+j)) + \cdots = \begin{cases} 0 & \text{if } n \text{ is odd} \\ q(n/2) & \text{if } n \text{ is even} \end{cases},$$

where $q(n)$ denotes the number of partitions of n into distinct parts.

Notation of Ramanujan

$$
\begin{aligned}
\psi(x) &:= 1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \cdots \\
&= \sum_{j \geq 0} x^{j(j+1)/2} \\
&= \sum_{j \in \mathbb{Z}} x^{2j^2 - j} \\
&= \prod_{k \geq 1} (1 - x^{4k})(1 + x^{2k-1}) \\
&= \prod_{k \geq 1} \frac{1 - x^{2k}}{1 - x^{2k-1}}.
\end{aligned}
$$

Proof of Triangular Number Recurrence

If for convenience we define $q(s) = 0$ when $s \notin \mathbb{Z}$,

$$\begin{aligned}
\sum_{n\ge 0} q(n/2)x^n = \sum_{m\ge 0} q(m)x^{2m} &= \prod_{k\ge 1}(1+x^{2k}) \\
&= \prod_{k\ge 1} \frac{1+x^k}{1+x^{2k-1}} \\
&= \prod_{k\ge 1} \frac{1}{1-x^k}\cdot\frac{1-x^{2k}}{1+x^{2k-1}} \\
&= P(x)\psi(-x) \\
&= \left(\sum_{k\ge 0} p(k)x^k\right)\left(\sum_{j\in\mathbb{Z}}(-1)^j x^{2j^2-j}\right) \\
&= \sum_{n\ge 0}\left(\sum_{j\in\mathbb{Z}}(-1)^j p\big(n-(2j^2-j)\big)\right)x^n.
\end{aligned}$$

Square Recurrence

Theorem

For all integers n,

$$p(n)-p(n-1)-p(n-2)+p(n-4)+p(n-8)-p(n-9)-p(n-18) \cdots + (-1)^j p(n-j^2) + (-1)^j p(n-2j^2) + \cdots = \begin{cases} 0 & \text{if } n \text{ is odd} \\ qq(n) & \text{if } n \text{ is even} \end{cases}, \qquad (7)$$

where $qq(n)$ denotes the number of partitions of n into distinct, odd parts.

Notation of Ramanujan

$$\begin{aligned}\varphi(x) &:= 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + 2x^{25} + 2x^{36} + \cdots \\ &= 1 + 2\sum_{j\geq 0} x^{j^2} \\ &= \sum_{j\in\mathbb{Z}} x^{j^2} \\ &= \prod_{k\geq 1} (1 - x^{2k})(1 + x^{2k-1})^2 \\ &= \prod_{k\geq 1} \frac{1 - x^k}{1 - (-x)^k}.\end{aligned}$$

Proof of Square Recurrence

Since

$$\sum_{n\geq 0} qq(n)x^n = \prod_{k\geq 1}(1 + x^{2k+1}),$$

and for any power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

it is the case that

$$\frac{f(x) + f(-x)}{2} = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots , \qquad (8)$$

Proof of Square Recurrence

$$\sum_{n\geq 0} \frac{1+(-1)^n}{2} qq(n)x^n = \sum_{k\geq 0} qq(2k)x^{2k}$$

$$= \frac{1}{2}\left(\prod_{k\geq 1}(1+x^{2k-1}) + \prod_{k\geq 1}(1-x^{2k-1}) \right)$$

$$= \frac{1}{2}\prod_{k\geq 1}\frac{1-x^{4k-2}}{1-x^{2k-1}} + \frac{1}{2}\prod_{k\geq 1}\frac{(1-x^{2k-1})(1-x^k)}{1-x^k}$$

$$= \frac{\varphi(-x^2)+\varphi(-x)}{2\prod_{k\geq 1}(1-x^k)}$$

$$= \left(\sum_{k\geq 0} p(k)x^k\right)\left(1+\sum_{j\geq 1}(-1)^j x^{2j^2} + \sum_{k\geq 1}(-1)^j x^{j^2}\right)$$

$$= \sum_{n\geq 0}\left(p(n)+\sum_{j\geq 1}(-1)^j\left[p(n-2j^2)+p(n-j^2)\right]\right)x^n.$$

Thank you!