MacMahon's partial fractions

Andrew Sills

Georgia Southern University

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Definitions

A *partition* λ of an integer n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ (for some l) of positive integers where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$.

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Let $\mathscr{P}(n)$ denote the set of partitions of *n*.

More definitions

A *weak m-composition* γ of n is an m-tuple $(\gamma_1, \ldots, \gamma_m)$ of nonnegative integers such that $\gamma_1 + \gamma_2 + \cdots + \gamma_m = n$.

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Let $p_m(n)$ denote the number of partitions of n of length at most m, i.e. the number of weak m-partitions of n.

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The *multipartition dissection* $M = M(\lambda)$ of the partition λ is the set of all $l(\lambda)$-tuples where the jth component of $M(\lambda)$ is a partition of λ_j, i.e.

$$M(\lambda) := \mathscr{P}_{\lambda_1} \times \mathscr{P}_{\lambda_2} \times \cdots \times \mathscr{P}_{\lambda_{l(\lambda)}},$$

where $\times$ denotes Cartesian product.

Example of multipartition dissection of a partition

$$M\Big((3,2)\Big) = \Big\{\Big((3),(2)\Big),\Big((2,1),(2)\Big),\Big((1,1,1),(2)\Big), \Big((3),(1,1)\Big),\Big((2,1),(1,1)\Big),\Big((1,1,1),(1,1)\Big)\Big\}.$$

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Let $\rho_k := \exp(2\pi i/k)$.

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Note: $\frac{17}{72} = \frac{1}{8} + \frac{1}{9}$ and $\frac{59}{288} = \frac{1}{9} + \frac{1}{16} + \frac{1}{32}$.

$$
\begin{aligned}
p_4(n) = \frac{17}{72} + \frac{59(n+1)}{288} + \frac{1}{8}\binom{n+2}{2} + \frac{1}{24}\binom{n+3}{3} \\
+ \frac{(-1)^n}{8} + \frac{(-1)^n(n+1)}{32} + \frac{3+i\sqrt{3}}{54}\omega^{2n} \\
+ \frac{3-i\sqrt{3}}{54}\omega^{n} + \frac{i^n}{16} - \frac{(-i)^n}{16},
\end{aligned}
$$

where $\omega = (1 + i\sqrt{3})/2$.

Percy Alexander MacMahon (1854–1929)

MacMahon's partial fraction decomposition

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MacMahon's partial fraction decomposition

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A: In the theory of the representation of the symmetric group $\mathfrak{S}_m$.

- Two permutations $\sigma, \tau \in \mathfrak{S}_m$ are *conjugate* iff $\exists \pi$ such that

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$$|K_\lambda| = \frac{m!}{1^{f_1} 2^{f_2} 3^{f_3} \cdots f_1! f_2! f_3! \cdots}$$

MacMahon's partial fraction decomposition

So we would like some formulation akin to

$$\sum_{\sigma \in \mathfrak{S}_m} \sigma \left(\frac{1}{(1-x)(1-x^2)\cdots(1-x^m)} \right) = \sum_{\lambda \in \mathscr{P}_m} \frac{|K_\lambda|}{(1-x)^{f_1}(1-x^2)^{f_2}(1-x^3)^{f_3}\cdots}.$$

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- We need to add subscripts to the variables, so that the symmetric group has something upon which to act.
- It would be really nice if, for each term of the RHS, those permutations in K_λ acted on the subscripted x's in such a way that the LHS = RHS. (This doesn't work.)

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Let us generalize the LHS to

$$\sum_{\sigma \in \mathfrak{S}_4} \sigma \left(\frac{1}{(1 - x_1)(1 - x_1 x_2)(1 - x_1 x_2 x_3)(1 - x_1 x_2 x_3 x_4)} \right).$$

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$$=\sum_{\sigma\in\mathfrak{S}_4}\sigma\left(\sum_{w_1\geq w_2\geq w_3\geq w_4\geq 0}x_1^{w_1}x_2^{w_2}x_3^{w_3}x_4^{w_4}\right)$$

$$=\sum_{\sigma\in\mathfrak{S}_4}\sigma\left(\sum_{w\in\mathscr{W}_4}x_1^{w_1}x_2^{w_2}x_3^{w_3}x_4^{w_4}\right)$$

$$=\sum_{\gamma\in\mathscr{C}_4}f_0(\gamma)!f_1(\gamma)!f_2(\gamma)!\cdots x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}x_4^{\gamma_4}.$$

Thus, we see that the LHS generates every weak m-composition (where the jth part appears as the exponent of x_j) exactly $f_0(\gamma)!f_1(\gamma)!f_2(\gamma)!\cdots$ times.

$$g\Big((4); x_1, x_2, x_3, x_4\Big) = \frac{1}{1 - x_1 x_2 x_3 x_4}$$

$$g\Big((3,1); x_1, x_2, x_3, x_4\Big) = \frac{1}{(1 - x_1 x_2 x_3)(1 - x_4)}$$

$$g\Big((2,2); x_1, x_2, x_3, x_4\Big) = \frac{1}{(1 - x_1 x_2)(1 - x_3 x_4)}$$

$$g\Big((2,1,1); x_1, x_2, x_3, x_4\Big) = \frac{1}{(1 - x_1 x_2)(1 - x_3)(1 - x_4)}$$

$$g\Big((1,1,1,1); x_1, x_2, x_3, x_4\Big) = \frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)}.$$

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- Permute the nonzero terms of F_γ into nonincreasing order to form a partition λ of weight $m = 4$, and we may write $\lambda = \lambda(\gamma)$, since the partition λ is uniquely determined by γ.
- Thus it must be the case that there exists $\sigma \in \mathfrak{S}_4$ such that the weak 4-composition $\sigma(\gamma)$ is of type $[c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_l^{\lambda_l}]$ for some distinct nonnegative integers $c_1, c_2, \dots, c_l$.

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Thus $\lambda(\gamma) = (3, 1)$.

$$M\big((3,1)\big) = \Big\{\big((3),(1)\big), \big((2,1),(1)\big), \big((1,1,1),(1)\big)\Big\}.$$

$$\sum_{\sigma\in\mathfrak{S}_4}\sigma\left(\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}\right)$$

$$=\frac{6}{1-x_1x_2x_3x_4}+2\left(\frac{1}{(1-x_1x_2x_3)(1-x_4)}+\frac{1}{(1-x_1x_2x_4)(1-x_3)}\right.$$

$$\left.+\frac{1}{(1-x_1x_3x_4)(1-x_2)}+\frac{1}{(1-x_1)(1-x_2x_3x_4)}\right)$$

$$+\left(\frac{1}{(1-x_1x_2)(1-x_3x_4)}+\frac{1}{(1-x_1x_3)(1-x_2x_4)}\right.$$

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$$+\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}.$$

RHS terms that generate $(5,5,5,13)$

$$\sum_{\sigma\in\mathfrak{S}_4}\sigma\left(\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}\right)$$

$$=\frac{6}{1-x_1x_2x_3x_4}+2\left(\frac{1}{(1-x_1x_2x_3)(1-x_4)}+\frac{1}{(1-x_1x_2x_4)(1-x_3)}\right.$$

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$$\left.+\frac{1}{(1-x_1)(1-x_2x_4)(1-x_3)}+\frac{1}{(1-x_1)(1-x_2)(1-x_3x_4)}\right)$$

$$+\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}.$$

λ	form of γ	generating terms of RHS
(4)	*aaaa*	$6g(4;\mathbf{x})+2\Big(()+(34)+(24)+(14)\Big)g(31;\mathbf{x})$ $+\Big(()+(23)+(24)\Big)g(22;\mathbf{x})$ $+\Big(()+(13)+(23)+(24)+(14)+(13)(24)\Big)g(211;\mathbf{x})$ $+g(1111;\mathbf{x})$
(31)	*aaab*	$2g(31;\mathbf{x})+\Big(()+(13)+(23)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	aaba	$2(34)g(31;\mathbf{x})+\Big(()+(24)+(14)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abaa	$2(24)g(31;\mathbf{x})+\Big((23)+(24)+(13)(24)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abbb	$2(14)g(31;\mathbf{x})+\Big((13)+(14)+(13)(24)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
(22)	*aabb*	$g(22;\mathbf{x})+\Big(()+(13)(24)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abab	$(23)g(22;\mathbf{x})+\Big((23)+(14)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abba	$(24)g(22;\mathbf{x})+\Big((13)+(24)\Big)g(211;\mathbf{x})+g(1111;\mathbf{x})$
(211)	*aabc*	$g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abac	$(23)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abca	$(24)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abbc	$(13)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abcb	$(14)g(211;\mathbf{x})+g(1111;\mathbf{x})$
	abcc	$(13)(24)g(211;\mathbf{x})+g(1111;\mathbf{x})$
(1111)	*abcd*	$g(1111;\mathbf{x})$

Table : The letters a, b, c, d represent distinct nonnegative integers. Permutations are presented in cycle notation.

Augustine O. Munagi

Munagi's q-partial fraction decomposition

$$\sum_{n=0}^{\infty} p_4(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

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$$\sum_{n=0}^{\infty} p_4(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

$$= \frac{25/144}{(1-x)^2} + \frac{1/8}{(1-x)^3} + \frac{1/24}{(1-x)^4} + \frac{1/16}{1-x^2} + \frac{1/8}{(1-x^2)^2} + \frac{(x+2)/9}{1-x^3} + \frac{1/4}{1-x^4}$$

Happy Birthday, George!