Identities of the Ramanujan-Slater type

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The Rogers-Ramanujan identities

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 1 \,(\mathrm{mod}\ 5)}} \frac{1}{1-q^j}$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2 \,(\mathrm{mod}\ 5)}} \frac{1}{1-q^j}$$

Integer Partitions

A *partition* λ of the integer n into ℓ parts is an ℓ-tuple

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

where

- each λ_i is a positive integer,
- $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_\ell$, and
- $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

ω The quantity n is called the *weight* of λ and is denoted $|\lambda|$. The number of parts of λ is called the *length* $\ell(\lambda)$ of λ.

There are seven partitions of 5:

$$(5) \quad (4,1) \quad (3,2) \quad (3,1,1) \quad (2,2,1)$$
$$(2,1,1,1) \quad (1,1,1,1,1)$$

First Rogers-Ramanujan identity—combinatorial version

The number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n where

$$\lambda_i - \lambda_{i+1} \geqq 2$$

for $i = 1, 2, \ldots, \ell - 1$

equals the number of partitions of n into parts congruent to $\pm 1 \pmod 5$.

Example: For $n = 10$, the relevant partitions are

$$(10) \quad (9,1) \quad (8,2) \quad (7,3) \quad (6,4) \quad (6,3,1)$$

$$(9,1) \quad (6,4) \quad (6,1,1,1,1) \quad (4,4,1,1) \quad (4,1,1,1,1,1,1)$$
$$(1,1,1,1,1,1,1,1,1,1)$$

a-generalized Rogers-Ramanujan identities

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \frac{1}{(aq;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j-1)/2}(1-aq^{2j})(a;q)_j}{(1-a)(q;q)_j}$$

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)} = \frac{1}{(aq;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j+3)/2}(1-aq^{2j+1})(aq;q)_j}{(q;q)_j},$$

where $(a;q)_j = (1-a)(1-aq)\cdots(1-aq^{j-1})$

and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$.

a-generalized Rogers-Ramanujan identities

$$F_2(a) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j-1)/2}(1-aq^{2j})(a;q)_j}{(1-a)(q;q)_j}$$

$$F_1(a) := \sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(1-q)(1-q^2)\cdots(1-q^j)}$$
$$= \frac{1}{(aq;q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j} q^{j(5j+3)/2}(1-aq^{2j+1})(aq;q)_j}{(q;q)_j}$$

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a-generalized Rogers-Ramanujan identities: associated q-difference equations

$$F_1(a) = F_2(aq)$$
$$F_2(a) = F_1(a) + aqF_2(aq)$$

Ramanujan recorded the following identity in the lost notebook:

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1-q^3)(1-q^9)\cdots(1-q^{6j-3})}{(1-q)^2(1-q^3)^2\cdots(1-q^{2j-1})^2(1-q^4)(1-q^8)\ldots(1-q^{4j})}$$
$$= \frac{1+q^2+q^6+\cdots}{1-q-q^3+\cdots}(1-q^6)(1-q^{18})\ldots$$

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-q)(1-q^3)\cdots(1-q^{2j-1})\times(1-q^4)(1-q^8)\ldots(1-q^{4j})}$$
$$= \prod_{\substack{j \geqq 1 \\ j \text{ odd or } j \equiv \pm 2 \pmod{12}}} \frac{1}{1-q^j}$$

$$\sum_{j=0}^{\infty} \frac{a^j q^{j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-aq)(1-aq^3)\cdots(1-aq^{2j-1})\times(1-q^4)(1-q^8)\ldots(1-q^{4j})}$$

$$= \prod_{j=1}^{\infty} \frac{1+aq^{4j-2}+a^2q^{8j-4}}{1-aq^{2j-1}}$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} s(\ell,n) a^{\ell} q^{n}$$

where $s(\ell, n)$ denotes the number of partitions of n with exactly ℓ parts where even parts may appear at most twice and are not multiples of 4.

Ramanujan-Slater Mod 8 Identity

$$\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\cdots(1+q^{2j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} = \prod_{\substack{j \geqq 1 \\ j \equiv \pm 1,4 \pmod 8}} \frac{1}{1-q^j}$$

First Göllnitz-Gordon partition identity

The number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n where

- $\lambda_i - \lambda_{i+1} \geqq 2$ for $i = 1, 2, \ldots, \ell - 1$,

- $\lambda_i - \lambda_{i+1} > 2$ if λ_i is even

equals the number of partitions of n into parts congruent to $\pm 1, 4 \pmod 8$.

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \frac{q^{j^2}(1+q)(1+q^3)\cdots(1+q^{2j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} \\
&= \sum_{j=0}^{\infty} \frac{q^{j^2} q(q^{-1}+1)q^3(q^{-3}+1)\cdots q^{2j-1}(q^{-(2j+1)}+1)}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} \\
&= \sum_{j=0}^{\infty} \frac{q^{j^2+1+3+\cdots+(2j-1)}(1+q^{-1})(1+q^{-3})\cdots(1+q^{-(2j+1)})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})} \\
&= \sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q^{-1})(1+q^{-3})\cdots(1+q^{-(2j+1)})}{(1-q^2)(1-q^4)\cdots(1-q^{2j})}
\end{aligned}
$$

Signed Partitions

A *signed partition* σ of the integer n is a partition pair (π, ν) where $|\pi| - |\nu| = n$.

- The positive parts of σ are the parts of π.
- The negative parts of σ are the parts of ν.

Andrews' variation on Göllnitz-Gordon for Signed Partitions:

The number of signed partitions $\sigma = (\pi, \nu)$ of n where

- all positive parts are even and at least $2\ell(\pi)$, and
- all negative parts are odd, distinct and less than $2\ell(\pi)$

equals the number of (ordinary) partitions of n into parts congruent to $\pm 1, 4 \pmod 8$.

Example: For $n = 10$, the relevant partitions are

$$(10) \quad (9,1) \quad (8,2) \quad (7,3) \quad (6,3,1)$$

$$\Big((10),\emptyset\Big) \quad \Big((6,4),\emptyset\Big) \quad \Big((10,4),(3,1)\Big) \quad \Big((8,6),(3,1)\Big)$$
$$\Big((6,6,6),(5,3)\Big)$$

$$(9,1) \quad (7,1,1,1) \quad (4,4,1,1) \quad (4,1,1,1,1,1,1)$$
$$(1,1,1,1,1,1,1,1,1,1)$$

The following identity was recorded by Ramanujan in the lost notebook:

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1-q^3)(1-q^9)\cdots(1-q^{6j-3})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})\times(1-q)(1-q^3)\ldots(1-q^{2j-1})}$$

$$= \frac{1-q-q^5+q^8+\cdots}{1-q-q^2+q^5+\cdots}(1-q^9)(1-q^{27})\cdots$$

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-1})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

$$= \prod_{\substack{j \geqq 1 \\ j \equiv \pm 2, \pm 3, \pm 4, \pm 8 (\text{mod } 18)}} \frac{1}{1-q^j}$$

In 1981, Andrews gave the following partition theoretic interpretation: Let $\rho_1(n)$ denote the number of partitions of n subject to the conditions:

- no part appears more than twice,
- no odd part exceeds the number of even parts,
- among the even parts (arranged in nonincreasing size) only the second, fourth, sixth, etc., may be subsequently repeated.

Let $\rho_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$.

Then $\rho_1(n) = \rho_2(n)$ for all n.

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2j-1}+q^{4j-2})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

$$= \sum_{j=0}^{\infty} \frac{q^{4j^2}(1+q^{-1}+q^{-2})\cdots(1+q^{-(2j-1)}+q^{-(4j-2)})}{(1-q^2)(1-q^4)\cdots(1-q^{4j})}$$

The number of signed partitions $\sigma = (\pi, \nu)$ of n where

- $\ell(\pi)$ is even, and each positive part is even and $\geqq \ell(\pi)$,
- the negative parts are odd, less than $\ell(\pi)$, and may appear at most twice

equals the number of partitions of n in parts congruent to $\pm 2, \pm 3, \pm 4, \pm 8 \pmod{18}$.

Mod 18 Identities—McLaughlin-S.

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} = \prod_{\substack{j\geqq 1 \\ j\equiv \pm 2,\pm 3,\pm 4,\pm 5,\pm 6 \pmod{18}}} \frac{1}{1-q^j}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-1;q^3)_n}{(-1;q)_n(q;q)_{2n}} = \prod_{\substack{j\geqq 1 \\ j\equiv \pm 1,\pm 3,\pm 4,\pm 6,\pm 8 \pmod{18}}} \frac{1}{1-q^j}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}} = \prod_{j\geqq 1} \frac{(1-q^{18j-3})(1-q^{18j-15})}{(1-q^{3j-1})(1-q^{3j-2})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3;q^3)_n}{(q^2;q^2)_n(q^{n+2};q)_{n+1}} = \prod_{\substack{j\geqq 1 \\ j\equiv \pm 2,\pm 3,\pm 6,\pm 7,\pm 8 \pmod{18}}} \frac{1}{1-q^j}$$

Rogers-Ramanujan and Lie Algebras

- Let $\mathfrak{g}$ be be affine Kac-Moody Lie Algebra $A_1^{(1)}$ or $A_2^{(2)}$.

- Let $\{h_0, h_1\}$ be the usual basis of a maximal toral sub-algebra T of $\mathfrak{g}$.

- Let d be the degree derivation of $\mathfrak{g}$.

- Let $\tilde{T} := T \oplus \mathbb{C}d$.

- For all dominant integral $\lambda \in \tilde{T}^*$, there is an essentially unique irreductible, integrable, highest weight module $L(\lambda)$ called the *standard module*, assuming WLOG that $\lambda(d) = 0$.

- $\lambda = s_0\Lambda_0 + s_1\Lambda_1$ where Λ_0 and Λ_1 are the fundamental weights, given by

$$\Lambda_i(h_j) = \delta_{ij} \text{ and } \Lambda_i(d) = 0.$$

- For $A_1^{(1)}$, the canonical central element is $c = h_0 + h_1$.

- For $A_2^{(2)}$, the canonical central element is $c = h_0 + 2h_1$.

- The quantity $\lambda(c)$ is called the *level* of $L(\lambda)$.

- There is an infinite product $F_{\mathfrak{g}}$ associated with $\mathfrak{g}$ called the "fudge factor."

- $\mathfrak{g}$ has a certain infinite-dimensional Heisenberg subalgebra known as the "principal Heisenberg vacuum subalgebra" $\mathfrak{s}$.

- The principal character $\chi(\Omega(s_0\Lambda_0+s_1\Lambda_1))$, where $\Omega(\lambda)$ is the vacuum space for $\mathfrak{s}$ in $L(\lambda)$ is
$$\chi(\Omega(s_0\Lambda_0+s_1\Lambda_1)) = \frac{\chi(L(s_0\Lambda_0+s_1\Lambda_1))}{F_{\mathfrak{g}}},$$
where $\chi(L(\lambda))$ is the principally specialized character of $L(\lambda)$.

In the case of $A_1^{(1)}$ for standard modules of odd level $2k+1$,

$$\chi(\Omega((2k-i+2)\Lambda_0+(i-1)\Lambda_1))$$
$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+3)j})(1-q^{(2k+3)j-i})(1-q^{(2k+3)j-2k-3+i})}{1-q^j}$$
$$=\sum_{n_1,n_2,\ldots,n_k\geqq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}},$$

where $1\leqq i\leqq k+1$ and $N_j:=n_j+n_{j+1}+\cdots+n_k$.

In the case of $A_1^{(1)}$ for standard modules of even level $2k$,

$$\chi(\Omega((2k-i+1)\Lambda_0+(i-1)\Lambda_1))$$
$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+2)j})(1-q^{(2k+2)j-i})(1-q^{(2k+2)j-2k-2+i})}{1-q^j}$$
$$=\sum_{n_1,n_2,\dots,n_k\geqq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_k^2+N_i+N_{i+1}+\cdots+N_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}(q^2;q^2)_{n_k}},$$

where $1\leqq i\leqq k+1$ and $N_j:=n_j+n_{j+1}+\cdots+n_k$.

Define

$$Q(w,x) := \prod_{j \geqq 1} (1-w^j)(1+xw^{j-1}) \left(1+\frac{w^j}{x}\right)\left(1-\frac{w^{2j-1}}{x^2}\right)(1-x^2w^{2j-1}).$$

The principal character for the level ℓ standard modules associated with $A_2^{(2)}$ is

$$\chi((\ell-2i+2)\Lambda_0+(i-1)\Lambda_1)) = \frac{Q(q^{\ell+3},-q^i)}{\prod_{j\geqq 1}(1-q^j)},$$

where $1 \leqq i \leqq 1+\lfloor\frac{\ell}{2}\rfloor$.

Principal characters of level 6 standard modules for $A_2^{(2)}$

$$\chi(6\Lambda_0) = \frac{Q(q^9, -q)}{\prod_{j \geqq 1} 1 - q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-1; q^3)_n}{(-1; q)_n (q; q)_{2n}}$$

$$\chi(4\Lambda_0 + \Lambda_1) = \frac{Q(q^9, -q^2)}{\prod_{j \geqq 1} 1 - q^j} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-1; q^3)_n}{(-1; q)_n (q; q)_{2n}}$$

$$\chi(2\Lambda_0 + 2\Lambda_1) = \frac{Q(q^9, -q^3)}{\prod_{j \geqq 1} 1 - q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^3; q^3)_n}{(-q; q)_n (q; q)_{2n+1}}$$

$$\chi(4\Lambda_1) = \frac{Q(q^9, -q^4)}{\prod_{j \geqq 1} 1 - q^j} = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q^3; q^3)_n}{(q^2; q^2)_n (q^{n+2}; q)_{n+1}}$$