On Dyson's *q*-Series Identities

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Analysis in the 19th century

- Periodic functions
- Doubly periodic functions
- Elliptic functions
- Theta Functions

Jacobi's Theta Function

$$\vartheta(z,q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$$

Ramanujan's theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$

Ramanujan's theta function

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$

$$= \prod_{j=0}^{\infty} (1 + a^{j+1} b^{j})(1 + a^{j} b^{j+1})(1 - a^{j+1} b^{j+1})$$

Ramanujan's theta function

$$f(-q,-q^2) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \ldots$$

Euler's
Pentagonal Number Theorem

$$f(-q,-q^2) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \ldots$$

$$= (1-q)(1-q^2)(1-q^3)(1-q^4)\ldots$$

Ramanujan's theta function

$$f(a,b) := \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} + \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}$$

Rogers's false theta function

$$\Psi(a,b) := \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} - \sum_{n=1}^{\infty} a^{n(n-1)/2} b^{n(n+1)/2}$$

Ramanujan's
mock theta functions

The Rogers-Ramanujan Identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)} = \frac{f(-q^2,-q^3)}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\ldots(1-q^n)} = \frac{f(-q,-q^4)}{f(-q,-q^2)}$$

The Rogers-Ramanujan Identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}$$

Dyson's mod 27 analogs of the RR identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(1+q+q^2)(1+q^2+q^4)\dots(1+q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\dots(1-q^{2n-1})\times(1-q^n)}=\frac{f(-q^{12},-q^{15})}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(1+q+q^2)(1+q^2+q^4)\dots(1+q^n+q^{2n})}{(1-q)(1-q^2)\dots(1-q^{2n+1})}=\frac{f(-q^9,-q^{18})}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(1+q+q^2)(1+q^2+q^4)\dots(1+q^n+q^{2n})}{(1-q)(1-q^2)\dots(1-q^{2n+2})}=\frac{f(-q^6,-q^{21})}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+3n}(1+q+q^2)(1+q^2+q^4)\dots(1+q^n+q^{2n})}{(1-q)(1-q^2)\dots(1-q^{2n+2})}=\frac{f(-q^3,-q^{24})}{f(-q,-q^2)}$$

Dyson's mod 27 analogs of the RR identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1+q+q^2)(1+q^2+q^4)\ldots(1+q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})} = \prod_{j=1}^{\infty} \frac{1-q^{9j}}{1-q^j}$$

Rogers-Ramanujan series

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}$$

Andrews' generalization of the Rogers-Ramanujan series

$$G(t,q) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^n)}$$

Andrews' generalization of the Rogers-Ramanujan series

$$G(t,q) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(1-t)(1-tq)(1-tq^2)\ldots(1-tq^n)}$$

$$G(t,q) = \frac{1}{1-t} + \frac{t^2 q}{1-t} G(tq,q)$$

$$G(t,q) = \sum_{N=0}^{\infty} D_N(q) t^N, \text{where } D_0(q) = D_1(q) = 1,$$

and $D_N(q) = D_{N-1}(q) + q^{N-1} D_{N-2}(q)$ for $N \geq 2$

Andrews' generalization of the Rogers-Ramanujan series

$$\lim_{t \to 1^-} (1-t) G(t,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}$$

Andrews' generalization of the Rogers-Ramanujan series

- This is one of Ramanujan's mock theta functions!

$$\lim_{t \to -1^-} (1-t)G(t,q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)(1+q^2)\ldots(1+q^n)}$$

Rogers's mod 10 identity

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(1-q)(1-q^2)\ldots(1-q^n)\times(1-q)(1-q^3)\ldots(1-q^{2n-1})} = \frac{f(-q^4,-q^6)}{f(-q,-q^2)}$$

Rogers's mod 10 identity

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(1-q)(1-q^2)\ldots(1-q^n)\times(1-q)(1-q^3)\ldots(1-q^{2n-1})}$$

$$=\prod_{j=1}^{\infty} \frac{(1-q^{10j-6})(1-q^{10j-4})(1-q^{10j})}{1-q^j}$$

Rogers's mod 10 series

$$R(q) = \sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(1-q)(1-q^2)\ldots(1-q^n) \times (1-q)(1-q^3)\ldots(1-q^{2n-1})}$$

Andrews' generalization of Rogers's mod 10 series

$$R(t,q)=\sum_{n=0}^{\infty}\frac{t^{3n}q^{n(3n-1)/2}}{(1-t)(1-tq)\dots(1-tq^{n})\times(1-t^{2}q)(1-t^{2}q^{3})\dots(1-t^{2}q^{2n-1})}$$

Andrews' generalization of Rogers's mod 10 series

$$\lim_{t \to -1}(1-t)R(t,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}}{(1+q)\ldots(1+q^n)\times(1-q)(1-q^3)\ldots(1-q^{2n-1})}$$

False theta series identity of Rogers

$$\lim_{t \to -1}(1-t)R(t,q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}}{(1+q)\ldots(1+q^n)\times(1-q)(1-q^3)\ldots(1-q^{2n-1})}$$

$$= \Psi(q^8,q^7) - q\Psi(q^2,q^{13})$$

Andrews' generalizations

- Sometimes lim as *t* approaches -1 gives a mock theta function.
- Sometimes lim as *t* approaches -1 gives a false theta function.
- There is another possibility. . .

Dyson's mod 27 identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(1+q+q^2)(1+q^2+q^4)\ldots(1+q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\ldots(1-q^{2n-1})\times(1-q^n)}=\frac{f(-q^{12},-q^{15})}{f(-q,-q^2)}$$

Mod 108 identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\ldots(1-q^{2n-1})\times(1+q^n)}$$

$$=\frac{f(-q^{12},-q^{15})-2q^2f(q^{33},q^{75})+2q^7f(q^{15},q^{93})}{f(-q,-q^2)}$$

Mod 108 identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\ldots(1-q^{2n-1})\times(1+q^n)}$$

$$=\frac{f(-q^{12},-q^{15})-2q^2f(q^{33},q^{75})+2q^7f(q^{15},q^{93})}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^{n}+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$=\frac{f(-q^{9},-q^{18})-2q^3f(q^{27},q^{81})+2q^9f(q^{9},q^{99})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+2})}$$

$$= \frac{f(-q^6,-q^{21}) - 2q^4 f(q^{21},q^{87}) + 2q^{11} f(q^3,q^{105})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+2})}$$

$$= \frac{f(-q^6,-q^{21}) - 2q^4 f(q^{21},q^{87}) + 2q^{11} f(q^3,q^{105})}{f(-q,-q^2)}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+3n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+2})}$$

$$= \frac{f(-q^3,-q^{24}) - 2q^5 f(q^{15},q^{93}) + 2q^{13} f(q^{-3},q^{111})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \frac{f(-q^9,-q^{18}) - 2q^3 f(q^{27},q^{81}) + 2q^9 f(q^9,q^{99})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \frac{f(q^{45},q^{63}) - q^9 f(q^9,q^{99}) - 2q^3 f(q^{27},q^{81}) + 2q^9 f(q^9,q^{99})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \frac{f(q^{45},q^{63}) + q^9 f(q^9,q^{99}) - 2q^3 f(q^{27},q^{81})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \frac{f(q^{45},q^{63}) + q^9 f(q^9,q^{99}) - q^3 f(1,q^{27})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \frac{f(q^9,q^{18}) - q^3 f(1,q^{27})}{f(-q,-q^2)}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{9j-6})(1-q^{9j-3})(1-q^{9j})(1-q^{18j-15})(1-q^{18j-3})}{1-q^j}$$

Mod 108 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\dots(1-q^n+q^{2n})}{(1-q)(1-q^2)\dots(1-q^{2n+1})}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{3j})(1-q^{18j-15})(1-q^{18j-3})}{1-q^j}$$

Mod 18 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+1})}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{3j})(1-q^{18j-15})(1-q^{18j-3})}{1-q^j}$$

Mod 18 identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\ldots(1-q^{2n})}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{9j-8})(1-q^{9j-1})(1-q^{9j})(1-q^{18j-11})(1-q^{18j-7})}{1-q^j}$$

Mod 18 identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^{n-1}+q^{2n-2})}{(1-q)(1-q^2)\ldots(1-q^{2n})}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{9j-7})(1-q^{9j-2})(1-q^{9j})(1-q^{18j-13})(1-q^{18j-5})}{1-q^j}$$

Mod 18 identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n})}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{9j-6})(1-q^{9j-3})(1-q^{9j})(1-q^{18j-15})(1-q^{18j-3})}{1-q^j}$$

Mod 18 identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(1-q+q^2)(1-q^2+q^4)\ldots(1-q^n+q^{2n})}{(1-q)(1-q^2)\ldots(1-q^{2n+2})}(1-q^{n+1})$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{9j-5})(1-q^{9j-4})(1-q^{9j})(1-q^{18j-17})(1-q^{18j-1})}{1-q^j}$$

Bailey Chains

- In the 1980's, Andrews discovered a way to iterate a formula of W.N. Bailey.
- Thus *every* Rogers-Ramanujan type identity implies infinitely many others.

Rising q-factorial notation

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$$

Bailey Chain associated with the 1st RR Identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=1}^{\infty} \frac{(1-q^{5j-3})(1-q^{5j-2})(1-q^{5j})}{1-q^j}$$

Bailey Chain associated with the 1st RR Identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{q^{(n_1+n_2)^2+n_2^2}}{(q;q)_{n_1}(q;q)_{n_2}}=\prod_{j=1}^{\infty}\frac{(1-q^{7j-4})(1-q^{7j-3})(1-q^{7j})}{1-q^j}$$

Bailey Chain associated with the 1st RR Identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3{}^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}}=\prod_{j=1}^{\infty}\frac{(1-q^{9j-5})(1-q^{9j-4})(1-q^{9j})}{1-q^{j}}$$

Lepowsky and Wilson

- provided the first Lie theoretic interpretation and proof of the Rogers-Ramanujan identities
- showed how identities in the RR Bailey chain corresponded to standard modules associated with the Lie algebra $A_1^{(1)}$

$A_1^{(1)}$ Level 1

$$1 = \prod_{j=1}^{\infty} \frac{(1-q^{3j-2})(1-q^{3j-1})(1-q^{3j})}{1-q^{j}}$$

$A_1^{(1)}$ Level 3

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{j=1}^{\infty} \frac{(1-q^{5j-3})(1-q^{5j-2})(1-q^{5j})}{1-q^j}$$

$A_1^{(1)}$ Level 5

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{q^{(n_1+n_2)^2+n_2^2}}{(q;q)_{n_1}(q;q)_{n_2}}=\prod_{j=1}^{\infty}\frac{(1-q^{7j-4})(1-q^{7j-3})(1-q^{7j})}{1-q^j}$$

$A_1^{(1)}$ Level 7

$$\sum_{n_1, n_2, n_3 \geq 0} \frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}} = \prod_{j=1}^{\infty} \frac{(1-q^{9j-5})(1-q^{9j-4})(1-q^{9j})}{1-q^j}$$

$A_1^{(1)}$ Level $2k$–1

$$\sum_{n_1,n_2,n_3,\ldots,n_{k-1}\geq 0} \frac{q^{(n_1+n_2+\ldots+n_{k-1})^2+(n_2+\ldots n_{k-1})^2+\ldots+(n_{k-2}+n_{k-1})^2+n_{k-1}{}^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}\ldots(q;q)_{n_{k-1}}}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+1)j-(k+1)})(1-q^{(2k+1)j-k})(1-q^{(2k+1)j})}{1-q^{j}}$$

$A_1^{(1)}$ Level 2

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} = \prod_{j=1}^{\infty} \frac{(1-q^{4j-2})(1-q^{4j-2})(1-q^{4j})}{1-q^j}$$

$A_1^{(1)}$ Level 4

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{q^{(n_1+n_2)^2+n_2^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}} = \prod_{j=1}^{\infty} \frac{(1-q^{6j-3})(1-q^{6j-3})(1-q^{6j})}{1-q^j}$$

$A_1^{(1)}$ Level 6

$$\sum_{n_1,n_2,n_3\geq 0}\frac{q^{(n_1+n_2+n_3)^2+(n_2+n_3)^2+n_3{}^2}}{(q;q)_{n_1}(q;q)_{n_2}(q^2;q^2)_{n_3}}=\prod_{j=1}^{\infty}\frac{(1-q^{8j-4})(1-q^{8j-4})(1-q^{8j})}{1-q^j}$$

$A_1^{(1)}$ Level $2k$

$$\sum_{n_1,n_2,n_3,\ldots,n_k\geq 0} \frac{q^{(n_1+n_2+\ldots+n_k)^2+(n_2+\ldots n_k)^2+\ldots+(n_{k-1}+n_k)^2+n_k{}^2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}\ldots(q;q)_{n_{k-1}}(q^2;q^2)_{n_k}}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{(2k+2)j-k})^2(1-q^{(2k+2)j})}{1-q^j}$$

$A_2^{(2)}$

$A_2^{(2)}$ Level 1

$$1=\prod_{j=1}^{\infty}\frac{(1-q^{4j-3})(1-q^{4j-1})(1-q^{4j})(1-q^{8j-2})(1-q^{8j-6})}{1-q^{j}}$$

$A_2^{(2)}$ Level 2

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^2)_n} = \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})(1-q^{5j-1})(1-q^{5j})(1-q^{10j-7})(1-q^{10j-3})}{1-q^j}$$

$A_2^{(2)}$ Level 3

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2n} \frac{q^{n^2}}{(q;q)_{2n-j}(q;q)_j} \left(\frac{n-j+1}{3}\right)$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{6j-5})(1-q^{6j-1})(1-q^{6j})(1-q^{12j-8})(1-q^{12j-4})}{1-q^j}$$

$A_2^{(2)}$ Level 4

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{n(n+1)/2+j}(-q)_n}{(q;q)_{2n}(q;q)_j}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{7j-4})(1-q^{7j-3})(1-q^{7j})(1-q^{14j-9})(1-q^{14j-5})}{1-q^j}$$

$A_2^{(2)}$ Level 4

$$\prod_{j=1}^{\infty}\frac{1}{1+q^j}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_{2n}}$$

$$=\prod_{j=1}^{\infty}\frac{(1-q^{7j-4})(1-q^{7j-3})(1-q^{7j})(1-q^{14j-9})(1-q^{14j-5})}{1-q^j}$$

$A_2^{(2)}$ Level 4

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_{2n}(-q^{n+1};q)_\infty}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{7j-4})(1-q^{7j-3})(1-q^{7j})(1-q^{14j-9})(1-q^{14j-5})}{1-q^j}$$

$A_2^{(2)}$ Level 5

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{j=1}^{\infty} \frac{(1-q^{8j-7})(1-q^{8j-1})(1-q^{8j})(1-q^{16j-10})(1-q^{16j-6})}{1-q^{j}}$$

$A_2^{(2)}$ Level 6

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-1;q^3)_n}{(-1;q)_n (q;q)_{2n}}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{9j-8})(1-q^{9j-1})(1-q^{9j})(1-q^{18j-11})(1-q^{18j-7})}{1-q^j}$$

$A_2^{(2)}$ Level 7

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}}$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{10j-9})(1-q^{10j-1})(1-q^{10j})(1-q^{20j-12})(1-q^{20j-8})}{1-q^{j}}$$