Disturbing the Dyson Conjecture. . .
. . . in a GOOD way

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Let n be a positive integer.

Dyson (1962) conjectured that the constant term in the expansion of the Laurent polynomial

$$\prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i}$$

is the multinomial coefficient

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!}.$$

Let $[x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}]X$ denote the coefficient of $x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$ in the expression X.

Thus, e.g.,

$$[x^2yz]\Big(4x^2yz - 15 + 2xyz\Big) = 4,$$

$$[x^3y^2](x+y)^5 = 10,$$

$$[x^0y^0z^0](3 + xyz) = 3.$$

The $n = 2$ case

$$[x_1^0 x_2^0]\left(1-\frac{x_2}{x_1}\right)^{a_1}\left(1-\frac{x_1}{x_2}\right)^{a_2} = \frac{(a_1+a_1)!}{a_1!a_2!}$$

is a consequence of the binomial theorem.

The $n = 3$ case

$$\begin{aligned}[x_1^0 x_2^0 x_3^0]\left(\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)\right)^{a_1}\left(\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_3}{x_2}\right)\right)^{a_2}\\ \times\left(\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\right)^{a_3}\\ = \frac{(a_1+a_2+a_3)!}{a_1!a_2!a_3!}\end{aligned}$$

is equivalent to a hypergeometric summation formula due to A. C. Dixon.

- Dyson proved $n = 4$ and $n = 5$ case.

- J. Gunson and K. Wilson independently proved Dyson's conjecture for general n in 1962.

- I. J. Good provided the most compact and elegant proof in 1970.

$$\mathbf{a} := \langle a_1, a_2, \ldots, a_n \rangle,$$

(*n*-vector of symbolic nonnegative integers)

$$\mathbf{x} := \langle x_1, x_2, \ldots, x_n \rangle,$$ (n-vector of indeterminants)

$$\mathbf{0} := \langle 0, 0, \ldots, 0 \rangle,$$ (n-dimensional zero vector)

$$\mathbf{e}_k := \langle 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0 \rangle,$$

(the n-vector with 1 in the kth position and 0 elsewhere)

$$\sigma_n(\mathbf{a}) := a_1 + a_2 + \cdots + a_n,$$

(first elementary symmetric polynomial in n indeterminants)

$$F_n(\mathbf{x}; \mathbf{a}) := \prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i}$$

(Dyson product)

$$c_n^{\mathbf{b}}(\mathbf{a}) := [x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}] F_n(\mathbf{x}; \mathbf{a})$$

Dyson's conjecture:

$$c_n^{\mathbf{0}}(\mathbf{a};\mathbf{x}) = \frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

Good's proof:

Recurrence. For $a_1, \ldots, a_n > 0$, by Lagrange interpolation,

$$F_n(\mathbf{x};\mathbf{a}) = \sum_{k=1}^{n} F_n(\mathbf{x};\mathbf{a}-\mathbf{e}_k).$$

Thus

$$c_n^{\mathbf{0}}(\mathbf{a}) = \sum_{k=1}^{n} c_n^{\mathbf{0}}(\mathbf{a}-\mathbf{e}_k). \qquad (R)$$

Initial condition.

$$c_n^{\mathbf{0}}(\mathbf{0}) = 1. \qquad (I)$$

Boundary conditions. For k fixed, $1 \leqq k \leqq n$,

$$F_n(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots a_n \rangle)$$
$$= F_{n-1}(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots a_n \rangle)$$
$$\times \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{(x_i - x_k)^{a_i}}{x_i^{a_i}}$$

$$[x_k^0] F_n(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots a_n \rangle)$$
$$= F_{n-1}(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots a_n \rangle)$$

Thus,

$$c_n^{\mathbf{0}}(\langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n \rangle)$$
$$= c_{n-1}^{\mathbf{0}}(\langle a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots a_n \rangle) \quad (B)$$

for $k = 1, 2, \ldots, n$.

- $c_n^{\mathbf{0}}(\mathbf{a})$ is uniquely determined by (R), (I), and (B).

- $\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}$ also satisfies (R), (I), and (B).

- Thus, $c_n^{\mathbf{0}}(\mathbf{a}) = \frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}$

$\square$

Natural question:
Can we find nice closed form representations for coefficients arising in the expansion of the Dyson product beside the constant term?

Reasonable conjecture:
For specific vectors $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$,

$$c_n^{\mathbf{b}}(\mathbf{a}) = [x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}] F_n(\mathbf{x}; \mathbf{a})$$
$$= \frac{\sigma_n(\mathbf{a})}{a_1! a_2! \cdots a_n!} \times \text{a rational function of the } a_i\text{'s}$$

$$
\begin{aligned}
&\left[\frac{x_1^2}{x_2 x_3}\right] F_3(\langle x_1, x_2, x_3\rangle; \langle a_1, a_2, a_3\rangle) \\
&= c_3^{\langle 2,-1,-1\rangle}(\langle a_1, a_2, a_3\rangle) \\
&= \frac{a_2 a_3 (2 + 2a_1 + a_2 + a_3)}{(1 + a_1 + a_2)(1 + a_1 + a_3)(1 + a_1)} \frac{(a_1 + a_2 + a_3)!}{a_1! a_2! a_3!}
\end{aligned}
$$

Let r, s, and t be distinct fixed integers with $1 \leqq r, s, t \leqq n$ and $n \geqq 3$.

$$\left[\frac{x_r}{x_s}\right] F_n(\mathbf{x}; \mathbf{a}) = -\left(\frac{a_s}{1 + \sigma_n(\mathbf{a}) - a_s}\right) \frac{\sigma_n(\mathbf{a})!}{a_1! a_2! \cdots a_n!}.$$

$$\left[\frac{x_r^2}{x_s x_t}\right] F_n(\mathbf{x}; \mathbf{a})$$
$$= \left(\frac{a_s a_t \Big((1 + \sigma_n(\mathbf{a})) + (1 + \sigma_n(\mathbf{a}) - a_s - a_t)\Big)}{(1 + \sigma_n(\mathbf{a}) - a_s - a_t)(1 + \sigma_n(\mathbf{a}) - a_s)(1 + \sigma_n(\mathbf{a}) - a_t)}\right)$$
$$\times \frac{\sigma_n(\mathbf{a})!}{a_1! a_2! \cdots a_n!}.$$

Let r, s, t, and u be distinct fixed integers with $1 \leqq r, s, t, u \leqq n$ and $n \geqq 4$.

$$\left[\frac{x_r x_s}{x_t x_u}\right] F_n(\mathbf{x}; \mathbf{a})$$
$$= \left(\frac{a_t a_u\Big((1+\sigma_n(\mathbf{a})) + (1+\sigma_n(\mathbf{a}) - a_t - a_u)\Big)}{(1+\sigma_n(\mathbf{a}) - a_t - a_u)(1+\sigma_n(\mathbf{a}) - a_t)(1+\sigma_n(\mathbf{a}) - a_u)}\right)$$
$$\times \frac{\sigma_n(\mathbf{a})!}{a_1! a_2! \cdots a_n!}.$$

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

(q-analog of the integer n)

$$(x;q)_k := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{k-1})$$

(rising q-factorial)

$$[n]_q! := \prod_{j=1}^{n} [j]_q = \frac{(q;q)_n}{(1-q)^n}$$

(q-factorial)

$$\mathcal{F}_n(\mathbf{x};\mathbf{a};q) := \prod_{1 \leqq i < j \leqq n} \left(\frac{x_i q}{x_j};q\right)_{a_j} \left(\frac{x_j}{x_i};q\right)_{a_i}$$

(q-Dyson product)

Andrews' q-Dyson Conjecture (1975)

The constant term in the expansion of

$$\prod_{1 \leqq i < j \leqq n} \left(\frac{x_i q}{x_j}; q \right)_{a_j} \left(\frac{x_j}{x_i}; q \right)_{a_i}$$

is the q-multinomial coefficient

$$\frac{[\sigma_n(\mathbf{a})]_q!}{[a_1]_q! [a_2]_q! \cdots [a_n]_q!}$$

$$= \frac{(q;q)_{\sigma_n(\mathbf{a})}}{(q;q)_{a_1} (q;q)_{a_2} \cdots (q;q)_{a_n}}$$

$$=: \begin{bmatrix} \sigma_n(\mathbf{a}) \\ a_1, a_2, \ldots, a_n \end{bmatrix}_q .$$

- The $n = 2$ case follows from the q-binoimal theorem.

- The $n = 3$ case is equivalent to a q-analog of Dixon's sum due to F. H. Jackson.

- The $n = 4$ case was proved by K.W.J. Kadell (1985) using an approach analogous to Good's.

- Good's proof does not generalize to the q-analog for general n.

The q-Dyson conjecture
a.k.a. the Zeilberger-Bressoud Theorem

- First proof: D. Zeilberger and D. Bressoud (1985—24 pages)

- Second proof: I. Gessel and G. Xin (2006—8 pages)

Conjecture 1.

Let r and s be fixed integers with $1 \leqq r \neq s \leqq n$ and $n \geqq 2$.

$$\left[\frac{x_r}{x_s}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q) \quad = \quad -\frac{q^{L(r,s)}[a_s]_q}{[1+\sigma_n(\mathbf{a})-a_s]_q} \begin{bmatrix} \sigma_n(\mathbf{a}) \\ a_1, a_2, \ldots, a_n \end{bmatrix}_q$$

where

$$L(r,s) = \begin{cases} 1+\sigma_n(\mathbf{a}) - \sum_{k=r}^{s} a_k, & \text{if } r < s \\ \sum_{k=s+1}^{r-1} a_k, & \text{if } r > s. \end{cases}$$

Conjecture 2.
Let r, s, and t be distinct fixed integers with $1 \leqq r, s, t \leqq n$ and $n \geqq 3$. WLOG $s < t$.

$$\left[x_r^2 x_s^{-1} x_t^{-1}\right] \mathcal{F}_n(\mathbf{x}; \mathbf{a}; q)$$
$$= \frac{q^{L(r,s,t)}[a_s]_q[a_t]_q\Big([1+\sigma_n(\mathbf{a})]_q + q^{M(r,s,t)}[1+\sigma_n(\mathbf{a})-a_s-a_t]_q\Big)}{[1+\sigma_n(\mathbf{a})-a_s-a_t]_q[1+\sigma_n(\mathbf{a})-a_s]_q[1+\sigma_n(\mathbf{a})-a_t]_q} \times \begin{bmatrix} \sigma_n(\mathbf{a}) \\ a_1, a_2, \ldots, a_n \end{bmatrix}_q,$$

where

$$L(r,s,t) = \begin{cases} 2 + 2\sigma_n(\mathbf{a}) - 2\sum_{k=r}^{t} a_k + \sum_{k=s+1}^{t-1} a_k, & \text{if } r < s < t, \\ 1 + \sigma_n(\mathbf{a}) - \sum_{k=s}^{t} a_k + 2\sum_{k=s+1}^{r-1} a_k, & \text{if } s < r < t, \\ 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=s+1}^{t-1} a_k, & \text{if } s < t < r, \end{cases}$$

and

$$M(r,s,t) = \begin{cases} a_t, & \text{if } r < s < t \text{ or } s < t < r, \\ a_s, & \text{if } s < r < t. \end{cases}$$

Conjecture 3.

Let r, s, t and u be distinct fixed integers with $1 \leqq r, s, t, u \leqq n$ and $n \geqq 4$. WLOG $r < s$ and $t < u$.

$$\left[\frac{x_r x_s}{x_t x_u}\right] \mathcal{F}_n(\mathbf{x}; \mathbf{a}; q)$$
$$= \frac{q^{L(r,s,t,u)}[a_t]_q[a_u]_q\Big([1+\sigma_n(\mathbf{a})]_q + q^{M(r,s,t,u)}[1+\sigma_n(\mathbf{a}) - a_t - a_u]_q\Big)}{[1+\sigma_n(\mathbf{a}) - a_t - a_u]_q[1+\sigma_n(\mathbf{a}) - a_t]_q[1+\sigma_n(\mathbf{a}) - a_u]_q}$$
$$\times \begin{bmatrix} \sigma_n(\mathbf{a}) \\ a_1, a_2, \ldots, a_n \end{bmatrix}_q,$$

where

$$L(r,s,t,u) = \begin{cases} 2+2\sigma_n(\mathbf{a}) - 2\sum_{k=r}^{u} a_k + \sum_{k=r}^{s-1} a_k + \sum_{k=t+1}^{u-1} a_k, & \text{if } r < s < t < u, \\ 1+\sigma_n(\mathbf{a}) - \sum_{k=r}^{u} a_k + \sum_{k=t+1}^{s-1} a_k, & \text{if } r < t < s < u, \\ 1+\sigma_n(\mathbf{a}) - \sum_{k=r}^{s-1} a_k + 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=t+1}^{u-1} a_k + 2\sum_{k=u+1}^{s-1} a_k, & \text{if } r < t < u < s, \\ 1+\sigma_n(\mathbf{a}) - \sum_{k=t}^{u} a_k + \sum_{k=r}^{s-1} a_k + 2\sum_{k=t+1}^{r-1} a_k, & \text{if } t < r < s < u, \\ \sum_{k=t+1}^{r-1} a_k + \sum_{k=u+1}^{s-1} a_k, & \text{if } t < r < u < s, \\ \sum_{k=r}^{s-1} a_k + \sum_{k=t+1}^{u-1} a_k + 2\sum_{k=u+1}^{r-1} a_k, & \text{if } t < u < r < s, \end{cases}$$

and

$$M(r,s,t,u) = \begin{cases} a_u, & \text{if } r < s < t < u \text{ or } r < t < u < s \text{ or } t < u < r < s, \\ 1+\sigma_n(\mathbf{a}) & \text{if } r < t < s < u \text{ or } t < r < u < s, \\ a_t, & \text{if } t < r < s < u. \end{cases}$$