A classical q-hypergeometric approach to $A_2^{(2)}$ standard modules

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Perfunctory slide defining the notation that probably everyone in the audience already knows

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Of course, q is just a SYMBOL.

More notation designed to intimidate any non-specialists present

The bilateral basic hypergeometric series is given by

$$ {}_t\psi_t \left[\begin{matrix} a_1, a_2, \ldots, a_t \\ b_1, b_2, \ldots, b_t \end{matrix} ; q, z \right] := \sum_{r \in \mathbb{Z}} \frac{(a_1, a_2, \ldots, a_t; q)_r}{(b_1, b_2, \ldots, b_t; q)_r} z^r. $$

Classical results

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Triple product identity (Jacobi)

$$\sum_{n\in\mathbb{Z}}(-1)^n z^n q^{n^2} = (q/z, zq, q^2; q^2)_\infty. \tag{JTP}$$

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Quintuple product identity (Fricke)

$$(-qz^3, -q^2z^{-3}, q^3; q^3)_\infty - z(-qx^{-3}, -q^2z^3, q^3; q^3)_\infty$$
$$= (q/z, z, q; q)_\infty (q/z^2, qz^2; q^2)_\infty. \qquad \text{(QPI)}$$

Bailey's ${}_6\psi_6$ sum

Theorem (Bailey, 1936)

$$ {}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{a^2q}{bcde}\right] = \frac{(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{be}, \frac{aq}{cd}, \frac{aq}{ce}, \frac{aq}{de}, q, \frac{q}{a}; q)_\infty}{(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{q}{b}, \frac{q}{c}, \frac{q}{d}, \frac{q}{e}, \frac{a^2q}{bcde}; q)_\infty} $$

provided $|a^2q/bcde| < 1$.

Bailey pairs, Bailey's lemma

If $(\alpha_n(a,q), \beta_n(a,q))$ satisfies

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

then (α_n, β_n) is called a *Bailey pair with respect to a*,

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then (α_n, β_n) is called a *Bailey pair with respect to a*, and $(\alpha'_n(a,q), \beta'_n(a,q))$ is also a Bailey pair, where

$$\alpha'_r(a,q) = \frac{(\rho_1)_r(\rho_2)_r}{(aq/\rho_1)_r(aq/\rho_2)_r} \left(\frac{aq}{\rho_1\rho_2}\right)^r \alpha_r$$

and

$$\beta'_n(a,q) = \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}}{(aq/\rho_1)_n(aq/\rho_2)_n(q)_{n-j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j(a,q).$$

“Weak” form of Bailey’s lemma

$$\sum_{n\geq 0} q^{n^2}\beta_n(1,q) = \frac{1}{(q)_\infty}\sum_{r\geq 0} q^{r^2}\alpha_r(1,q) \qquad \text{(WBL)}$$

The pair

$$\Big(\alpha_n(a,q),\beta_n(a,q)\Big) = \left(\frac{(-1)^n a^n q^{n(3n-1)/2}(1-aq^{2n})(a)_n}{(1-a)(q)_n}, \frac{1}{(q)_n} \right)$$

is a Bailey pair with respect to a.

A particular Bailey pair

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Insert this pair into (WBL):

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{r\geq 0} (-1)^r q^{r(5r-1)/2}(1+q^r)$$

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is a Bailey pair with respect to a.
Rewrite RHS as a bilateral series:

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n} = \frac{1}{(q)_\infty}\sum_{r\in\mathbb{Z}}(-1)^r q^{r(5r-1)/2}$$

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is a Bailey pair with respect to a.
Apply (JTP):

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q)_\infty}$$

The Rogers–Ramanujan identities

Theorem (Rogers, 1894)

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$$\sum_{n\geq 0} \frac{q^{n(n+1)}}{(q)_n} = \frac{(q, q^4, q^5; q^5)_\infty}{(q)_\infty}.$$

143rd time that "partition" is being defined at this conference

A *partition* λ of n is a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers (called the *parts* of λ) that sum to n.

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The seven partitions of 5 are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

Rogers–Ramanujan identities: partition theoretic version

- The number of partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n where

$$\lambda_j - \lambda_{j+1} \geq 2 \text{ for } j = 1, 2, \ldots, l(\lambda) - 1$$

equals the number of partitions of n into parts $\equiv \pm 1 \pmod{5}$.

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equals the number of partitions of n into parts $\equiv \pm 1 \pmod 5$.
- The number of partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n where
$$\lambda_j > 1 \text{ for } j = 1, 2, \ldots, l(\lambda)$$
$$\lambda_j - \lambda_{j+1} \geq 2 \text{ for } j = 1, 2, \ldots, l(\lambda) - 1$$
equals the number of partitions of n into parts $\equiv \pm 2 \pmod 5$.

B. Gordon's combinatorial generalization of RR (1961)

Let k be a positive integer and $1 \leq i \leq k$.

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Let $B_{k,i}(n)$ denote the number of partitions λ of n where

- at most $i-1$ of the parts of λ equal 1,
- $\lambda_j - \lambda_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, l(\lambda) + 1 - k$.

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Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.

Note: The case $k = 2$ gives the standard combinatorial interpretation of the two RR identities.

Andrews' analytic counterpart to Gordon's theorem

Theorem (Andrews, 1974)

$$\sum_{n_{k-1}\geq n_{k-2}\geq\cdots\geq n_1\geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+n_{i+1}+\cdots+n_{k-1}}}{(q)_{n_1}(q)_{n_2-n_1}(q)_{n_3-n_2}\cdots(q)_{n_{k-1}-n_{k-2}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}.$$

Connections to Lie algebras

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- The two Rogers–Ramanujan identities occur at level 3.
- The even levels of $A_1^{(1)}$ correspond to D. Bressoud's even modulus analog of Andrews–Gordon.

Capparelli's identities (1988)

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Theorem (Conjectured: Capparelli, 1988; Proved: Andrews, 1992)

The number of partitions of n into parts $\equiv \pm 2, \pm 3 \pmod{12}$ *equals the number of partitions* $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ *of n where*

- $\lambda_i - \lambda_{i+1} \geq 2$,
- $\lambda_i - \lambda_{i+1} = 2 \implies \lambda_i \equiv 1 \pmod 3$,
- $\lambda_i - \lambda_{i+1} = 3 \implies \lambda_i \equiv 0 \pmod 3$

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The number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ equals the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_l)$ of n where

- $\lambda_i - \lambda_{i+1} \geq 2$
- $\lambda_i - \lambda_{i+2} \geq 3$
- $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,
- $\lambda_i - \lambda_{i+2} = 3$ and $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$.
- $\lambda_i - \lambda_{i+2} = 4$ and $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$,
- Consider the first differences $\Delta\lambda := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{l-1} - \lambda_l)$. None of the following subwords are permitted in $\Delta\lambda$: $(3,3,0), (3,2,3,0), (3,2,2,3,0), \dots, (3,2,2,2,2,\dots,2,3,0)$.

Lepowsky and Wilson's VOA perspective

Let $\mathfrak{g}$ denote the affine Kac–Moody Lie algebra $A_1^{(1)}$ or $A_2^{(2)}$. Let h_0, h_1 denote the usual basis of a maximal toral subalgebra T of $\mathfrak{g}$. Let d denote the *degree derivation* of $\mathfrak{g}$ and $\tilde{T} := T \oplus \mathbb{C}d$. For all dominant integral $\lambda \in \tilde{T}^*$, there is an essentially unique irreducible, integrable, highest weight module $L(\lambda)$, assuming without loss of generality that $\lambda(d) = 0$. Now $\lambda = s_0\Lambda_0 + s_1\Lambda_1$ where Λ_0 and Λ_1 are the fundamental weights, given by $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$; here s_0 and s_1 are nonnegative integers.

Lepowsky and Wilson's VOA perspective

For $A_1^{(1)}$, the canonical central element is $c = h_0 + h_1$, while for $A_2^{(2)}$, the canonical central element is $c = h_0 + 2h_1$. The quantity $\lambda(c)$ (which equals $s_0 + s_1$ for $A_1^{(1)}$ and which equals $s_0 + 2s_1$ for $A_2^{(2)}$) is called the *level* of $L(\lambda)$.For brevity, we may refer to $L(\lambda) = L(s_0\Lambda_0 + s_1\Lambda_1)$ as the "(s_0, s_1)-module."

Additionally, there is an infinite product $F_{\mathfrak{g}}$ associated with $\mathfrak{g}$, sometimes called the *fudge factor*, which needs to be divided out of the the principally specialized character $\chi(L(\lambda)) = \chi(s_0\Lambda_0 + s_1\Lambda_1)$, in order to obtain the quantities of interest here. For $\mathfrak{g} = A_1^{(1)}$, the fudge factor is given by $F_{\mathfrak{g}} = (q;q^2)_\infty^{-1}$, while for $\mathfrak{g} = A_2^{(2)}$, it is given by $F_{\mathfrak{g}} = \left[(q;q^6)_\infty (q^5;q^6)_\infty\right]^{-1}$.

Now $\mathfrak{g}$ has a certain infinite-dimensional Heisenberg subalgebra known as the *principal Heisenberg vacuum subalgebra* $\mathfrak{s}$ The principal character $\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1))$, where $\Omega(\lambda)$ is the vacuum space for $\mathfrak{s}$ in $L(\lambda)$, is

$$\chi(\Omega(s_0\Lambda_0 + s_1\Lambda_1)) = \frac{\chi(L(s_0\Lambda_0 + s_1\Lambda_1))}{F_{\mathfrak{g}}},$$

where $\chi(L(\lambda))$ is the principally specialized character of $L(\lambda)$.

Principal character of level ℓ standard modules of $A_1^{(1)}$

$$\chi(\Omega((\ell - i + 1)\Lambda_0 + (i-1)\Lambda_1)) = \frac{(q^i, q^{\ell+1-i}, q^{\ell+2}; q^{\ell+2})}{(q)_\infty},$$

where $1 \leqq i \leqq 1 + \lfloor \frac{\ell}{2} \rfloor$

Principal character of level ℓ standard modules of $A_2^{(2)}$

$$\chi(\Omega((\ell-2i+2)\Lambda_0+(i-1)\Lambda_1)) = \frac{(q^i, q^{\ell+3-i}, q^{\ell+3}; q^{\ell+3})_\infty (q^{\ell+3-2i}, q^{\ell+2i+3}; q^{2\ell+6})_\infty}{(q)_\infty},$$

where $1 \leqq i \leqq 1 + \lfloor \frac{\ell}{2} \rfloor$.

Bailey pairs for $A_2^{(2)}$

Let $(\alpha_n^{(\ell,i)}, \beta_n^{(\ell,i)})$ denote the Bailey pair which, upon insertion into (WBL) with $a = 1$, gives the principally specialized character of the $A_2^{(2)}$ standard module

$$\chi(\Omega((\ell - 2i + 2)\Lambda_0 + (i-1)\Lambda_1)).$$

Bailey pairs for $\chi(\Omega(\ell\Lambda_0))$

$$\alpha_n^{(\ell,1)}(1,q) = \begin{cases} 1 & \text{if } n=0 \\ q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r}(1+q^{(\ell-3)r}) & \text{if } n=3r>0 \\ -q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell-3)r} & \text{if } n=3r+1 \\ -q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell-3)r} & \text{if } n=3r-1 \end{cases}$$

$$
\begin{aligned}
\beta_n^{(\ell,1)}(1,q) &= \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,1)}(1,q)}{(q)_{n-s}(q)_{n+s}} \\
&= \frac{q^n}{(q)_n(q^2;q)_n} \\
&\qquad \times \sum_{r\in\mathbb{Z}} \frac{(1-q^{6r+1})(q^{-n};q)_{3r}}{(1-q)(q^{n+2};q)_{3r}} (-1)^r q^{\frac{3}{2}(\ell-6)r^2+\frac{1}{2}(\ell-6)r}.
\end{aligned}
\tag{1}
$$

For each $\ell = 1, 2, 3, \ldots$, the series expression in (1) is a limiting case of a very-well-poised bilateral basic hypergeometric series.

$$
\frac{q^{-n}(q)_n(q)_{n+1}}{1-q}\beta^{(\ell,1)}(1,q)
= \begin{cases}
\lim_{e\to 0} {}_8\psi_8\left[\begin{matrix} q^{\frac72},-q^{\frac72},q^{-n},q^{1-n},q^{2-n},e,e,e \\ q^{\frac12},-q^{\frac12},q^{n+4},q^{n+3},q^{n+2},\frac{q^4}{e},\frac{q^4}{e},\frac{q^4}{e}\end{matrix};q^3,\frac{q^{3n+6}}{e^3}\right] & \text{if } \ell=3 \\
\lim_{e\to 0} {}_8\psi_8\left[\begin{matrix} q^{\frac72},-q^{\frac72},q^{-n},q^{1-n},q^{2-n},e,e,-q^2 \\ q^{\frac12},-q^{\frac12},q^{n+4},q^{n+3},q^{n+2},\frac{q^4}{e},\frac{q^4}{e},-q^2\end{matrix};q^3,\frac{-q^{3n+4}}{e^2}\right] & \text{if } \ell=4 \\
\lim_{e\to 0} {}_6\psi_6\left[\begin{matrix} q^{\frac72},-q^{\frac72},q^{-n},q^{1-n},q^{2-n},e \\ q^{\frac12},-q^{\frac12},q^{n+4},q^{n+3},q^{n+2},\frac{q^4}{e}\end{matrix};q^3,\frac{q^{3n+2}}{e}\right] & \text{if } \ell=5 \\
{}_6\psi_6\left[\begin{matrix} q^{\frac72},-q^{\frac72},q^{-n},q^{1-n},q^{2-n},-q^2 \\ q^{\frac12},-q^{\frac12},q^{n+4},q^{n+3},q^{n+2},-q^2\end{matrix};q^3,-q^{3n}\right] & \text{if } \ell=6 \\
\lim_{e\to \infty} {}_6\psi_6\left[\begin{matrix} q^{\frac72},-q^{\frac72},q^{-n},q^{1-n},q^{2-n},e \\ q^{\frac12},-q^{\frac12},q^{n+4},q^{n+3},q^{n+2},\frac{q^4}{e}\end{matrix};q^3,\frac{q^{3n+2}}{e}\right] & \text{if } \ell=7
\end{cases}.
$$

Levels 5, 6, 7

$$\beta_n^{(5,1)}(1,q) = \frac{q^{n^2}}{(q)_{2n}},$$

$$\beta_n^{(6,1)}(1,q) = \frac{q^n(-1;q^3)_n}{(-1;q)_n(q)_{2n}},$$

$$\beta_n^{(7,1)}(1,q) = \frac{q^n}{(q)_{2n}}.$$

To evaluate β_n for levels $\ell = 3, 4, 8, 9$, we can use the following identity analogous to Bailey's ${}_6\psi_6$ sum: for nonnegative integer n,

$$
{}_8\psi_8 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, c, d, e, f, aq^{-n}, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, q^{n+1}, aq^{n+1} \end{matrix} ; q, \frac{a^2 q^{2n+2}}{cdef} \right]
= \frac{(aq, \frac{q}{a}, \frac{aq}{cd}, \frac{aq}{ef}; q)_n}{(\frac{q}{c}, \frac{q}{d}, \frac{aq}{e}, \frac{aq}{f}; q)_n} {}_4\psi_4 \left[\begin{matrix} e, f, \frac{aq^{n+1}}{cd}, q^{-n} \\ \frac{aq}{c}, \frac{aq}{d}, q^{n+1}, \frac{ef}{aq^n} \end{matrix} ; q, q \right] .
\tag{8psi8Trans}
$$

If ℓ is even and $\ell \geqq 6$,

$$q^{-n}(q)_n(q^2;q)_{n+1}\beta_n^{(\ell,1)}(1,q)$$

$$= \lim_{e\to\infty} {}_\ell\psi_\ell\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, \overbrace{e, e, \dots, e}^{\ell-6}, -q^2 \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \underbrace{\frac{q^4}{e}, \frac{q^4}{e}, \dots, \frac{q^4}{e}}_{\ell-6}, -q^2 \end{matrix}; q^3, \frac{-q^{3n+2\ell-12}}{e^{\ell-6}}\right],$$

while if ℓ is odd and $\ell > 6$,

$$q^{-n}(q)_n(q^2;q)_{n+1}\beta_n^{(\ell,1)}(1,q)$$

$$= \lim_{e\to\infty} {}_{\ell-1}\psi_{\ell-1}\left[\begin{matrix} q^{\frac{7}{2}}, -q^{\frac{7}{2}}, q^{-n}, q^{1-n}, q^{2-n}, \overbrace{e, e, \dots, e}^{\ell-6} \\ q^{\frac{1}{2}}, -q^{\frac{1}{2}}, q^{n+4}, q^{n+3}, q^{n+2}, \underbrace{\frac{q^4}{e}, \frac{q^4}{e}, \dots, \frac{q^4}{e}}_{\ell-6} \end{matrix}; q^3, \frac{q^{3n+2\ell-12}}{e^{\ell-6}}\right].$$

Then, to obtain the series and product expressions for $\chi(\Omega(\ell\Lambda_0))$, one inserts the Bailey pair $\left(\alpha_n^{(\ell,1)}(1,q), \beta_n^{(\ell,1)}(1,q)\right)$ into (WBL) with $a = 1$, and upon applying (JTP) and (QPI), we find that

$$\sum_{m=0}^{\infty} q^{9m^2}\left(q^{-6m+1}\beta_{3m-1}^{(\ell,1)}(1,q)+\beta_{3m}^{(\ell,1)}(1,q)+q^{6m+1}\beta_{3m+1}^{(\ell,1)}(1,q)\right)$$
$$= \frac{(q, q^{\ell+2}, q^{\ell+3}; q^{\ell+3})_\infty (q^{\ell+1}, q^{\ell+5}; q^{2\ell+6})}{(q)_\infty}.$$

Bailey pairs for $\chi(\Omega((\ell-2)\Lambda_0+\Lambda_1))$

$$\alpha_n^{(\ell,2)}(1,q) = \begin{cases} 1 & \text{if } n=0 \\ q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(9-\ell)r}(1+q^{(9-\ell)r}) & \text{if } n=3r>0 \\ -q^{\frac{3}{2}(\ell-3)r^2+\frac{1}{2}(\ell+3)r+1} & \text{if } n=3r+1 \\ -q^{\frac{3}{2}(\ell-3)r^2-\frac{1}{2}(\ell+3)r+1} & \text{if } n=3r-1 \end{cases}$$

$$\beta_n^{(\ell,2)}(1,q) = \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,2)}(1,q)}{(q)_{n-s}(q)_{n+s}}$$

$$= \frac{1}{(q)_n(q^2;q)_n} \sum_{r\in\mathbb{Z}} \frac{(1-q^{6r+1})(q^{-n};q)_{3r}}{(1-q)(q^{n+2};q)_{3r}} (-1)^r q^{\frac{3}{2}(\ell-6)r^2+\frac{1}{2}(\ell-6)r}.$$

$$\beta_n^{(\ell,2)}(1,q) = \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,2)}(1,q)}{(q)_{n-s}(q)_{n+s}}$$

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Notice that

$$q^n \beta_n^{(\ell,2)}(1,q) = \beta_n^{(\ell,1)}(1,q).$$

$$\beta_n^{(\ell,2)}(1,q) = \sum_{s=0}^{n} \frac{\alpha_s^{(\ell,2)}(1,q)}{(q)_{n-s}(q)_{n+s}}$$

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Notice that

$$q^n \beta_n^{(\ell,2)}(1,q) = \beta_n^{(\ell,1)}(1,q).$$

And so it follows that the series and product expressions for $\chi(\Omega((\ell-2)\Lambda_0+\Lambda_1))$ are

$$\sum_{m=0}^{\infty} q^{9m^2}\left(q^{-6m+1}\beta_{3m-1}^{(\ell,1)}(1,q)+\beta_{3m}^{(\ell,2)}(1,q)+q^{6m+1}\beta_{3m+1}^{(\ell,2)}(1,q)\right)$$

$$= \frac{(q^2,q^{\ell+1},q^{\ell+3};q^{\ell+3})_\infty(q^{\ell-1},q^{\ell+7};q^{2\ell+6})}{(q)_\infty}.$$

Level 3: $\chi(\Omega(3\Lambda_0))$

By direct substitution in (8psi8Trans), we find

$$\beta_{3m}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}},$$

By direct substitution in (8psi8Trans), we find

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$$\beta_{3m+1}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{7}{2}r+1}(q^2;q^3)_r}{(q^2;q^3)_{2m+1}(q^3;q^3)_{2m-r}(q)_{3r+1}}.$$

Level 3: $\chi(\Omega(3\Lambda_0))$

By direct substitution in (8psi8Trans), we find

$$\beta^{(3,1)}_{3m}(1,q) = \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac32 r^2 + \frac12 r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}},$$

$$\beta^{(3,1)}_{3m+1}(1,q) = \sum_{r=0}^{2m} \frac{(-1)^r q^{\frac32 r^2 + \frac72 r+1}(q^2;q^3)_r}{(q^2;q^3)_{2m+1}(q^3;q^3)_{2m-r}(q)_{3r+1}}.$$

$$\beta^{(3,1)}_{3m-1}(1,q) = ???.$$

For convenience, let us define the abbreviation

$$\sigma(m,r) := \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}},$$

For convenience, let us define the abbreviation

$$\sigma(m,r) := \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}},$$

so that we have immediately

$$\beta_{3m}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \sigma(m,r),$$

For convenience, let us define the abbreviation

$$\sigma(m,r) := \frac{(-1)^r q^{\frac{3}{2}r^2+\frac{1}{2}r}(q^2;q^3)_r}{(q^2;q^3)_{2m}(q^3;q^3)_{2m-r}(q)_{3r}},$$

so that we have immediately

$$\beta_{3m}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \sigma(m,r),$$

and with a bit of elementary algebra,

$$\beta_{3m+1}^{(3,1)}(1,q) = \sum_{r=0}^{2m} \frac{\sigma(m,r)}{1-q^{6m+2}} \left(\frac{1}{1-q^{3r+1}} - 1 \right),$$

for $m \geqq 0$.

q-Zeilberger algorithm to the rescue!

From the Paule–Riese `qZeil.m` *Mathematica* package one can find that $\beta_n^{(3,1)}(1,q)$ satisfies the recurrence

$$\beta_n = \frac{-q^2 + q^{2n} + q^{2n+1}}{q^2(1-q^{2n})(1-q^{2n-1})}\beta_{n-1} - \frac{1}{(1-q^{2n})(1-q^{2n-1})}\beta_{n-2}$$

q-Zeilberger algorithm to the rescue!

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as certified by the rational function

$$\frac{q^{-n-6r-2}\left(q^{3r}-q^n\right)\left(q^{3r+1}-q^n\right)\left(q^{3r+2}-q^n\right)}{\left(q^n-1\right)\left(q^n+1\right)\left(q^{2n}-q\right)\left(q^{6r+1}-1\right)}.$$

Thus

$$\beta_{3m-1}(1,q) = \sum_{r=0}^{2m} \sigma(m,r) \left(q^{6m} - \frac{1-q^{6m+1}}{1-q^{3r+1}} \right),$$

for $m \geqq 1$.

Inserting $\left(\alpha_n^{(3,1)}(1,q), \beta_n^{(3,1)}(1,q)\right)$ into (WBL) with $a = 1$, and applying (JTP) and (QPI), we find that

$$\sum_{m=0}^{\infty} q^{9m^2} \left(q^{-6m+1} \beta^{(3,1)}_{3m-1}(1,q) + \beta^{(3,1)}_{3m}(1,q) + q^{6m+1} \beta^{(3,1)}_{3m+1}(1,q) \right)$$

$$\sum_{m=0}^{\infty} \sum_{r=0}^{2m} q^{9m^2} \sigma(m,r) \left(q^{1-6m} \left(q^{6m} - \tfrac{1-q^{6m+1}}{1-q^{3r+1}} \right) + 1 + \tfrac{q^{6m+1}}{1-q^{6m+2}} \left(1 - \tfrac{1}{1-q^{3r+1}} \right) \right)$$

$$= \frac{(q, q^5, q^6; q^6)_\infty (q^4, q^8; q^{12})_\infty}{(q)_\infty} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty}$$

$$= (-q^2; q^2)_\infty (-q^3; q^6)_\infty.$$

Alladi, Andrews, Gordon (1995)

$$\sum_{i,j,k\geq 0} \frac{q^{3i^2+i+3j^2-j+3k(k+1)/2+3ik+3jk}(-q^3;q^3)_{i+j}}{(q^6;q^6)_i(q^6;q^6)_j(q^3;q^3)_k} = \frac{1}{(q^2,q^3,q^9,q^{10};q^{12})_\infty}$$

S. (2004)

$$\sum_{n=0}^{\infty} \sum_{j=0}^{2n} \frac{q^{n^2} \left(\frac{n-j+1}{3} \right)}{(q)_{2n-j}(q)_j} = \frac{1}{(q^2, q^3, q^9, q^{10}; q^{12})_\infty},$$

where $\left(\frac{n}{3}\right)$ is the Legendre symbol.

Thank you for listening!