q-Pell Sequences and Two Identities of V. A. Lebesgue

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Abstract

We examine a pair of Rogers-Ramanujan type identities of V. A. Lebesgue, and give polynomial identities for which the original identities are limiting cases. The polynomial identities turn out to be q-analogs of the Pell sequence. Finally, we provide combinatorial interpretations for the identities.

Key words: partitions, q-series, combinatorial identities, Pell numbers
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1 Introduction

The following two series-product identities were published by V. A. Lebesgue [11] in 1840. See also Andrews [5].

$$\sum_{j=0}^{\infty} \frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=0}^{\infty} \frac{1+q^{2j+1}}{1-q^{2j+1}} \tag{1.1}$$

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$$\sum_{j=0}^{\infty} \frac{(-q;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=1}^{\infty} \frac{1-q^{4j}}{1-q^j}, \tag{1.2}$$

where

$$(a;q)_j = \prod_{k=0}^{j-1}(1-aq^k)$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Slater reproved these identities and included them in her list of 130 identities of the Rogers-Ramanujan type [13, p. 152, eqns. (12) and (8) respectively].

In order to gain increased combinatorial insight, we will give finite versions of (1.1) and (1.2), i.e. polynomial identities for which (1.1) and (1.2) are limiting cases. It turns out that q-trinomial coefficients, which were discovered by Andrews and Baxter [7], are central to understanding the polynomial identities we provide.

In section 2, we provide a brief introduction to q-trinomial coefficients, along with statements of results which we will require in subsequent sections. In section 3, we present and prove the polynomial identities for which (1.1) and (1.2) are limiting cases, and show how (1.1) and (1.2) follow as corollaries of the polynomial identites. In section 4, we provide combinatorial interpretations of the polynomial identities in the case $q = 1$. In section 5, we prove partition theorems related to the Lebesgue identities.

2 Background Material

Consider the polynomial $(1+x+x^2)^n$. By expansion, we find

$$(1+x+x^2)^n = \sum_{j=-\infty}^{\infty} \binom{n}{j}_2 x^{j+n} \tag{2.1}$$

where

$$\binom{n}{j}_2 = \sum_{h \geqq 0} (-1)^h \binom{n}{h} \binom{2n-2h}{n-j-h}. \tag{2.2}$$

These $\binom{n}{j}_2$ are called *trinomial coefficients*, (not to be confused with the coefficients which arise in the expansion of $(x+y+z)^n$, which are also often called trinomial coefficients).

In contrast to the binomial coefficients which seem to have only one useful q-analog, namely the Gaussian polynomials $\left[{A \atop B}\right]_q$, the trinomial coefficients have several important q-analogs, of which two will be required for our present purposes:

$$\mathrm{T}_{0}(m,A,q) := \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}_q \tag{2.3}$$

$$\mathrm{T}_{1}(m,A,q) := \sum_{j=0}^{m} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}_q \tag{2.4}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := \begin{cases} (q;q)_A (q;q)_B^{-1} (q;q)_{A-B}^{-1}, & \text{if } 0 \leqq B \leqq A \\ 0 & \text{otherwise.} \end{cases}$$

Note: The definitions (2.3) and (2.4) are due to Andrews and Baxter [7, p. 299, eqns (2.8) and (2.9)]

The following Pascal triangle type relationship is easily deduced from (2.1):

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2. \tag{2.5}$$

We will require the following q-analogs of (2.5), which are due to Andrews and Baxter [7, pp. 300-1, eqns. (2.16) and (2.19) respectively]: For $m \geqq 1$,

$$\mathrm{T}_{1}(m,A,q) = \mathrm{T}_{1}(m-1,A,q) + q^{m+A}\mathrm{T}_{0}(m-1,A+1,q) + q^{m-A}\mathrm{T}_{0}(m-1,A-1,q) \tag{2.6}$$

$$\mathrm{T}_{0}(m,A,q) = \mathrm{T}_{0}(m-1,A-1,q) + q^{m+A}\mathrm{T}_{1}(m-1,A,q) + q^{2m+2A}\mathrm{T}_{0}(m-1,A+1,q) \tag{2.7}$$

The following identity of Andrews and Baxter [7, p. 301, eqn. (2.20)], which reduces to the tautology "$0 = 0$" in the case where $q = 1$ is also useful:

$$\mathrm{T}_{1}(m,A,q) - q^{m-A}\mathrm{T}_{0}(m,A,q) - \mathrm{T}_{1}(m,A+1,q) + q^{m+A+1}\mathrm{T}_{0}(m,A+1,q) = 0. \tag{2.8}$$

From (2.1), it is easy to deduce the symmetry relationship

$$\binom{n}{-j}_2 = \binom{n}{j}_2. \tag{2.9}$$

Two q-analogs of (2.9) are

$$\mathrm{T}_{0}(m,-A,q) = \mathrm{T}_{0}(m,A,q) \tag{2.10}$$

and

$$\mathrm{T}_{1(m,-A,q)} = \mathrm{T}_{1(m,A,q)}. \tag{2.11}$$

Note also the asymptotics of $\mathrm{T}_{1(m,A,q)}$ (Andrews and Baxter [7, p. 310, eqn. (2.51)]):

$$\lim_{m\to\infty} \mathrm{T}_{1(m,A,q)} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.12}$$

We will use the following corollaries of the q-binomial theorem; see e.g. Andrews, Askey, and Roy [6, p. 488, Cor. 10.2.2], or Andrews [2, p. 36, Theorem 3.3].

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)/2} t^j = (t;q)_n, \tag{2.13}$$

$$\sum_{j=0}^{\infty} \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q t^j = \frac{1}{(t;q)_n}. \tag{2.14}$$

We also require Jacobi's Triple Product Identity (see, e.g. Andrews [2, p. 21, Theorem 2.8] or Andrews, Askey, and Roy [6, p. 497, Thm. 10.4.1]): For $z \neq 0$ and $|q| < 1$,

$$\sum_{j=-\infty}^{\infty} z^j q^{j^2} = (-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty. \tag{2.15}$$

3 Finite versions of a pair of Series-Product Identities of Lebesgue

We restate the Lebesgue identites (1.1) and (1.2):

$$\sum_{j=0}^{\infty} \frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=0}^{\infty} \frac{1+q^{2j+1}}{1-q^{2j+1}} \tag{1.1}$$

$$\sum_{j=0}^{\infty} \frac{(-q;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=1}^{\infty} \frac{1-q^{4j}}{1-q^j} \tag{1.2}$$

In [12, p. 64, eqn. (6.2)], Santos conjectured a finite form of (1.1), which we now present as a theorem. (Note that this identity is deducible from an identity of Berkovich, McCoy and Orrick [9, p. 805, eqn. (2.34)].)

Theorem 1 *For all nonnegative integers n,*

$$\sum_{j=0}^{n}\sum_{k=0}^{j} q^{(j^2+j+k^2-k)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = \sum_{j=-n-1}^{n+1} (-1)^j q^{2j^2} \mathrm{T}_1(n+1, 4j+1, \sqrt{q}). \tag{3.1}$$

PROOF. Let $P_n(q)$ denote the polynomial on the LHS of (3.1). Let

$$f(q,t) := \sum_{n=0}^{\infty} P_n(q) t^n.$$

Then

$$\begin{aligned} f(q,t) &= \sum_{n=0}^{\infty} t^n \sum_{j=0}^{n}\sum_{k=0}^{j} q^{(j^2+j+k^2-k)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q \\ &= \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q^{\binom{j+1}{2}+\binom{k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} t^{j+k+l} q^{\binom{j+1}{2}+\binom{k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} l+j \\ j \end{bmatrix}_q \\ &= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (-t)^k \right\} \left\{ \sum_{l=0}^{\infty} \begin{bmatrix} j+l \\ l \end{bmatrix}_q t^l \right\} q^{j(j+1)/2} t^j \\ &= \sum_{j=0}^{\infty} \frac{(-t;q)_j t^j q^{j(j+1)/2}}{(t;q)_{j+1}} \quad \text{by (2.13) and (2.14).} \end{aligned}$$

Also,

$$\begin{aligned} f(q,t) &= \frac{1}{1-t} + \sum_{j=1}^{\infty} \frac{(-t;q)_j t^j q^{j(j+1)/2}}{(t;q)_{j+1}} \\ &= \frac{1}{1-t} + \sum_{j=0}^{\infty} \frac{(-t;q)_{j+1} t^{j+1} q^{(j+1)(j+2)/2}}{(t;q)_{j+2}} \\ &= \frac{1}{1-t} + \frac{(1+t)(tq)}{1-t} f(q,tq). \end{aligned}$$

Thus,

$$(1-t)f(q,t) = 1 + (tq+t^2q)f(q,tq),$$

and so

$$P_0(q) = 1, \qquad P_1(q) = 1+q, \tag{3.2}$$

$$P_n(q) - (1+q^n)P_{n-1}(q) - q^{n-1}P_{n-2}(q) = 0, \text{for } n \geqq 2. \tag{3.3}$$

Therefore, in order to prove the theorem, it is sufficient to show that the RHS of (3.1) satisfies the initial conditions (3.2) and the recurrence (3.3).

We shall demonstrate that the RHS of (3.1) satisfies the recurrence (3.3) by substituting it into the LHS of (3.3) and showing that it simplifies to 0.

$$\begin{aligned}&\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\mathrm{T}_1(n+1,4j+1,\sqrt{q})-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\mathrm{T}_1(n,4j+1,\sqrt{q})\\&-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_1(n,4j+1,\sqrt{q})-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_1(n-1,4j+1,\sqrt{q})\end{aligned}$$

Expand the first term by (2.6):

$$\begin{aligned}=\;&\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\mathrm{T}_1(n,4j+1,\sqrt{q})+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{n}{2}+1}\mathrm{T}_0(n,4j+2,\sqrt{q})\\&+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{n}{2}}\mathrm{T}_0(n,4j,\sqrt{q})-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}\mathrm{T}_1(n,4j+1,\sqrt{q})\\&-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_1(n,4j+1,\sqrt{q})-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_1(n-1,4j+1,\sqrt{q})\end{aligned}$$

In the above, the first term cancels the fourth. Next, apply (2.10) to the second term, then apply (2.7) to the second and third terms and (2.6) to the fifth:

$$\begin{aligned}=&\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{n}{2}+1}\mathrm{T}_0(n-1,-4j-3,\sqrt{q})+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_1(n-1,-4j-2,\sqrt{q})\\&+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{3}{2}n-1}\mathrm{T}_0(n-1,-4j-1,\sqrt{q})+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{n}{2}}\mathrm{T}_0(n-1,4j-1,\sqrt{q})\\&+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_1(n-1,4j,\sqrt{q})+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{3}{2}n}\mathrm{T}_0(n-1,4j+1,\sqrt{q})\\&-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_1(n-1,4j+1,\sqrt{q})+\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{3}{2}n+\frac{1}{2}}\mathrm{T}_0(n-1,4j+2,\sqrt{q})\\&-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{3}{2}n-\frac{1}{2}}\mathrm{T}_0(n-1,4j,\sqrt{q})-\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_1(n-1,4j+1,\sqrt{q})\end{aligned}$$

The fifth, sixth, seventh, and ninth terms sum to 0 by (2.8). Also by (2.8), the second, third, and eighth terms sum to $q^{2j^2+n}\mathrm{T}_{1(n-1,4j+1,\sqrt{q})}$, leaving

$$\begin{aligned}
= \quad & \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{n}{2}+1}\mathrm{T}_{0(n-1,-4j-3,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{n}{2}}\mathrm{T}_{0(n-1,4j-1,\sqrt{q})} \\
& - \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_{1(n-1,4j+1,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_{1(n-1,4j+1,\sqrt{q})}
\end{aligned}$$

In the above, replace j by $-j$ in the first term, then expand the first and second terms by (2.7) and the third and fourth terms by (2.6):

$$\begin{aligned}
= & \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{n}{2}+1}\mathrm{T}_{0(n-2,2-4j,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-4j+n+2}\mathrm{T}_{1(n-2,3-4j,\sqrt{q})} \\
& + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-6j+\frac{3}{2}n+3}\mathrm{T}_{0(n-2,4-4j,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{n}{2}}\mathrm{T}_{0(n-2,4j-2,\sqrt{q})} \\
& + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_{1(n-2,4j-1,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{3}{2}n-2}\mathrm{T}_{0(n-2,4j,\sqrt{q})} \\
& - \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n-1}\mathrm{T}_{1(n-2,4j+1,\sqrt{q})} - \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{3}{2}n-1}\mathrm{T}_{0(n-2,4j+2,\sqrt{q})} \\
& - \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{3}{2}n-2}\mathrm{T}_{0(n-2,4j,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+n}\mathrm{T}_{1(n-2,4j+1,\sqrt{q})} \\
& + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2+2j+\frac{3}{2}n}\mathrm{T}_{0(n-2,4j+2,\sqrt{q})} + \sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2-2j+\frac{3}{2}n-1}\mathrm{T}_{0(n-2,4j,\sqrt{q})}
\end{aligned}$$

After replacing j by $j+1$ in the second term and third terms, the second term cancels the tenth, and the third cancels the twelfth. After replacing j by $-j$ and applying (2.11) to the fifth term, the fifth cancels the seventh. Also, the sixth cancels the ninth:

$$
\begin{aligned}
&= \sum_{j=-\infty}^{\infty} (-1)^j (1+q) q^{2j^2-2j+\frac{n}{2}} \mathrm{T}_{0(n-2,4j-2,\sqrt{q})} \\
&\quad + \sum_{j=-\infty}^{\infty} (-1)^j (1-q) q^{2j^2+2j+\frac{3}{2}n-1} \mathrm{T}_{0(n-2,4j+2,\sqrt{q})} \\
&= \sum_{j=-\infty}^{\infty} (-1)^j (1+q+q^{n-1}-q^n) q^{2j^2+2j+\frac{n}{2}} \mathrm{T}_{0(n-2,4j+2,\sqrt{q})} \\
&= \sum_{j\,\mathrm{even}(=2k)} (-1)^j (1+q+q^{n-1}-q^n) q^{2j^2+2j+\frac{n}{2}} \mathrm{T}_{0(n-2,4j+2,\sqrt{q})} \\
&\quad + \sum_{j\,\mathrm{odd}(=2k-1)} (-1)^j (1+q+q^{n-1}-q^n) q^{2j^2+2j+\frac{n}{2}} \mathrm{T}_{0(n-2,4j+2,\sqrt{q})} \\
&= \sum_{k=-\infty}^{\infty} (1+q+q^{n-1}-q^n) q^{8k^2+4k+\frac{n}{2}} \mathrm{T}_{0(n-2,8k+2,\sqrt{q})} \\
&\quad - \sum_{k=-\infty}^{\infty} (1+q+q^{n-1}-q^n) q^{8k^2-4k+\frac{n}{2}} \mathrm{T}_{0(n-2,8k-2,\sqrt{q})} \\
&= 0.
\end{aligned}
$$

The above string of equations, combined with the easily checked initial conditions (3.2)

$$\sum_{j=-1}^{1} (-1)^j q^{2j^2} \mathrm{T}_{1(1,4j+1,\sqrt{q})} = 1 = P_0(q)$$

$$\sum_{j=-2}^{2} (-1)^j q^{2j^2} \mathrm{T}_{1(2,4j+1,\sqrt{q})} = 1+q = P_1(q)$$

establishes the theorem. $\square$

Note that the first Lebesgue identity (1.1) now follows as an easy corollary of Theorem 1.

Corollary 2 *For $|q| < 1$,*

$$\sum_{j=0}^{\infty} \frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=0}^{\infty} \frac{1+q^{2j+1}}{1-q^{2j+1}}.$$

PROOF. First, we take the limit of the RHS of (3.1).

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{j=-n-1}^{n+1} (-1)^j q^{2j^2} T_1(n+1, 4j+1, \sqrt{q}) \\
&= \lim_{n\to\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} T_1(n+1, 4j+1, \sqrt{q}) \\
&= (q^2;q^4)_\infty^2 (q^4;q^4)_\infty \lim_{n\to\infty} \mathrm{T}_{1(n+1,4j+1,\sqrt{q})} \quad \text{by (2.15)} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} (q^2;q^4)_\infty^2 (q^4;q^4)_\infty \quad \text{by (2.12)} \\
&= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.
\end{aligned}
$$

Next, we take the limit of the LHS of (3.1):

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{j=0}^{n} \sum_{k=0}^{j} q^{(j^2+j+k^2-k)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q \\
&= \lim_{n\to\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q^{j(j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q q^{k(k-1)/2} \begin{bmatrix} n-k \\ j \end{bmatrix}_q \\
&= \lim_{n\to\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q^{j(j+1)/2} q^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \frac{(q;q)_{n-k}}{(q;q)_j (q;q)_{n-j-k}} \\
&= \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2}}{(q;q)_j} \sum_{k=0}^{\infty} q^{k(k-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \\
&= \sum_{j=0}^{\infty} \frac{(-1;q)_j q^{j(j+1)/2}}{(q;q)_j} \qquad \text{by (2.13).}
\end{aligned}
$$

□

Next, we present a finite form of (1.2). Note that this identity is a special case an identity from statistical mechanics due to Berkovich, McCoy, and Orrick [9, p. 805, eqn. (2.34), with $L = n+1$, $\nu = 2$, $s' = 1$, and $r' = i = 0$].

Theorem 3 *For all nonnegative integers n,*

$$
\sum_{j=0}^{n} \sum_{k=0}^{j} q^{(j^2+j+k^2+k)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = \sum_{j=-n-1}^{n+1} (-1)^j q^{2j^2+j} \mathrm{T}_{1(n+1,4j+1,\sqrt{q})}. \tag{3.4}
$$

PROOF. Let $Q_n(q)$ denote the polynomial on the LHS of (3.4). Let

$$
g(q,t) := \sum_{n=0}^{\infty} Q_n(q) t^n.
$$

Then

$$\begin{aligned}
g(q,t) &= \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q^{\binom{j+1}{2}+\binom{k+1}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} t^{j+k+l} q^{\binom{j+1}{2}+\binom{k+1}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} l+j \\ j \end{bmatrix}_q \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} (-tq)^k \right\} \left\{ \sum_{l=0}^{\infty} \begin{bmatrix} j+l \\ l \end{bmatrix}_q t^l \right\} q^{j(j+1)/2} t^j \\
&= \sum_{j=0}^{\infty} \frac{(-tq;q)_j t^j q^{j(j+1)/2}}{(t;q)_{j+1}} \quad \text{by (2.13) and (2.14)}
\end{aligned}$$

Also,

$$\begin{aligned}
g(q,t) &= \frac{1}{1-t} + \sum_{j=1}^{\infty} \frac{(-tq;q)_j t^j q^{j(j+1)/2}}{(t;q)_{j+1}} \\
&= \frac{1}{1-t} + \sum_{j=0}^{\infty} \frac{(-tq;q)_{j+1} t^{j+1} q^{(j+1)(j+2)/2}}{(t;q)_{j+2}} \\
&= \frac{1}{1-t} + \frac{(1+tq)(tq)}{1-t} g(q,tq)
\end{aligned}$$

Thus,

$$(1-t)g(q,t) = 1 + (tq + t^2q^2)g(q,tq),$$

and so

$$Q_0(q) = 1, \qquad Q_1(q) = 1+q \tag{3.5}$$

$$Q_n(q) - (1+q^n)Q_{n-1}(q) - q^n Q_{n-2}(q) = 0 \tag{3.6}$$

So, in order to prove the theorem, it is sufficient to show that the RHS of (3.4) satisfies the initial conditions (3.5) and the recurrence (3.6).

We will demonstrate that the RHS of (3.4) satisfies the recurrence (3.6) by substituting it into the LHS of (3.3) and showing that it simplifies to 0.

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \mathrm{T}_{1(n+1,4j+1,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \mathrm{T}_{1(n,4j+1,\sqrt{q})} \\
&- \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n,4j+1,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,4j+1,\sqrt{q})}
\end{aligned}$$

In the above, expand the first term by (2.6):

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \mathrm{T}_{1(n,4j+1,\sqrt{q})} + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+\frac{n}{2}+1} \mathrm{T}_{0(n,4j+2,\sqrt{q})}$$
$$+ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+\frac{n}{2}} \mathrm{T}_{0(n,4j,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \mathrm{T}_{1(n,4j+1,\sqrt{q})}$$
$$- \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n,4j+1,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,4j+1,\sqrt{q})}$$

In the above, the first term cancels the fourth. Apply (2.10) to the second term, then expand the second and third terms by (2.7) and the fifth term by (2.6):

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+\frac{n}{2}+1} \mathrm{T}_{0(n-1,-4j-3,\sqrt{q})} + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,-4j-2,\sqrt{q})}$$
$$+ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+\frac{3}{2}n-1} \mathrm{T}_{0(n-1,-4j-1,\sqrt{q})} + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+\frac{n}{2}} \mathrm{T}_{0(n-1,4j-1,\sqrt{q})}$$
$$+ \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,4j,\sqrt{q})} + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+\frac{3}{2}n} \mathrm{T}_{0(n-1,4j+1,\sqrt{q})}$$
$$- \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,4j+1,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+\frac{3}{2}n+\frac{1}{2}} \mathrm{T}_{0(n-1,4j+2,\sqrt{q})}$$
$$- \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+\frac{3}{2}n-\frac{1}{2}} \mathrm{T}_{0(n-1,4j,\sqrt{q})} - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+n} \mathrm{T}_{1(n-1,4j+1,\sqrt{q})}$$

In the above, the second, third, seventh, and eighth sum to 0 by (2.8). Likewise, by (2.8), the fifth, sixth, ninth, and tenth sum to 0. After applying (2.10) to the first term, we have

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+\frac{n}{2}+1} \mathrm{T}_{0(n-1,4j+3,\sqrt{q})} + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+\frac{n}{2}} \mathrm{T}_{0(n-1,4j-1,\sqrt{q})} \tag{3.7}$$

which can be seen to

$$= 0$$

after shifting j to $j+1$ in the second term.

The above string of equations, combined with the easily checked initial conditions (3.5), establish the theorem. □

The second Lebesgue identity (1.2) now follows as an easy corollary of Theorem 3. The proof is similar to that of Corollary 2, therefore we choose to omit it.

Corollary 4 *For* $|q| < 1$,

$$\sum_{j=0}^{\infty} \frac{(-q;q)_j q^{j(j+1)/2}}{(q;q)_j} = \prod_{j=1}^{\infty} \frac{1-q^{4j}}{1-q^j}.$$

4 A combinatorial interpretation of the Pell sequence

Following the notation of the proof of Theorem 1, we have established, by setting $q = 1$ in (3.1), (3.2) and (3.3), that

$$P_n(1) = \sum_{j=-n-1}^{n+1} (-1)^j \binom{n+1}{4j+1}_2$$

and also that

$$P_0(1) = 1 \qquad P_1(1) = 2$$
$$P_n(1) = 2P_{n-1}(1) + P_{n-2}(1), \quad n \geqq 2.$$

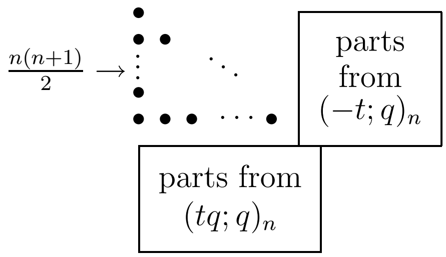
Thus, $\{P_n(1)\}_{n=0}^{\infty} = \{1, 2, 5, 12, 29, 70, 169, \dots\}$, the Pell sequence. Equivalently, $\{P_n(q)\}_{n=0}^{\infty}$ is a q-analog of the Pell sequence.

In order to find a combinatorial interpretation for $P_n(1)$, we write

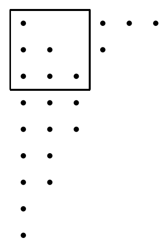
$$\begin{aligned} f(q,t) &= \sum_{n=0}^{\infty} P_n(q) t^n \\ &= \sum_{n=0}^{\infty} \frac{(-t;q)_n t^n q^{n(n+1)/2}}{(t;q)_{n+1}} \\ &= \frac{1}{1-t} \sum_{n=0}^{\infty} \frac{(-t;q)_n q^{n(n+1)/2}}{(tq;q)_n} \end{aligned}$$

and define a Modified Frobenius Symbol[1] (MFS) in which on the top row we may have at most one copy of each number from the set $\{0, 1, 2, \dots, n-1\}$ (in decreasing order from left to right). This means that on the top row we may have some (or even all) entries empty. An empty entry is represented by a dash ($-$). The bottom row of the MFS contains at least $n-1$ and at most n positive integers in decreasing order. Actually, we are representing a partition in the following form:

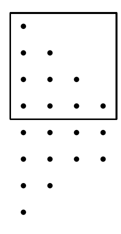
[1] For a further discussion of Frobenius symbols, see [4] and [3].



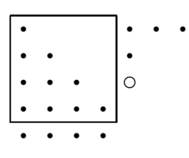
To illustrate this, we list three different partitions of 22 together with the corresponding graphical representation:



$$\begin{pmatrix} 3 & 1 & - \\ 8 & 5 & 2 \end{pmatrix}$$



$$\begin{pmatrix} - & - & - & - \\ 8 & 5 & 3 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 3 & 1 & 0 & - \\ 5 & 4 & 3 & 2 \end{pmatrix}$$

One can see that the exponent of t^N in

$$\frac{(-t;q)_n t^n q^{n(n+1)/2}}{(tq;q)_n}$$

is the number of parts plus the number of nonzero entries on the top row of the MFS.

Hence, $P_n(q)$ is the generating function for partitions in which the number of parts plus the number of nonzero entries on the top row of the MFS is no more than n. Considering that $P_n(1)$ is the sequence of Pell numbers $\{1, 2, 5, 12, 29, \dots\}$ we have proved the following

Theorem 5 *The total number of partitions in which the number of parts plus the number of nonzero entries on the top row of the MFS does not exceed n equals*

$$P_n(1) = \sum_{j=-n-1}^{n+1} (-1)^j \binom{n+1}{4j+1}_2.$$

Using the notation of the proof of Theorem 3, we see that $\{Q_n(q)\}_{n=0}^{\infty}$ is also a q-analog of the Pell sequence. Analogous to the preceeding, we define a new modified Frobenius symbol MFS', which is the same as MFS, except that the entries of the top row are chosen from the set $\{1, 2, 3, \ldots, n\}$, rather than from the set $\{0, 1, 2, \ldots, n-1\}$ and obtain the following

Theorem 6 *The total number of partitions in which the number of parts plus the number of nonzero entries on the top row of the MFS' does not exceed n equals*

$$Q_n(1) = P_n(1) = \sum_{j=-n-1}^{n+1} (-1)^j \binom{n+1}{4j+1}_2.$$

5 Combinatorics of the Lebesgue Identities

We give partition theoretic interpretations of the two Lebesgue Identities in terms of two-color partitions. In two-color partition theory, one has two copies of the integers (suppose one copy is blue and the other copy white). For example, the ten unrestricted two-color partitions of 3 are

$$3_b, \quad 3_w, \quad 2_b + 1_b, \quad 2_b + 1_w, \quad 2_w + 1_b, \quad 2_w + 1_w,$$

$$1_b + 1_b + 1_b, \quad 1_b + 1_b + 1_w, \quad 1_b + 1_w + 1_w, \quad 1_w + 1_w + 1_w,$$

where the subscripts indicate the color of the part.

Two parts will be called *distinct* if they are of different color and/or different magnitude. Two parts will be called *numerically distinct* only if they are of different magnitude.

Posession of a polynomial identity which converges to a series-product identity is a valuable asset in the quest for partition identities. Let us examine what we can do with

$$\sum_{j=0}^{n} \sum_{k=0}^{j} q^{(j^2+j+k^2-k)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q = \sum_{j=-n-1}^{n+1} (-1)^j q^{2j^2} \mathrm{T}_{1(n+1,4j+1,\sqrt{q})}. \qquad (3.1)$$

Since we have a recurrence formula for the polynomials $P_n(q)$ in (3.1), with the aid of a computer algebra package, we can generate $P_n(q)$ for small values of n. Also, to make the partitions being generated as transparent as possible, rather than expressing the recurrence in the form

$$P_0 = 1, \qquad P_1 = 1 + q \tag{3.2}$$
$$P_n = (1 + q^n)P_{n-1} + q^{n-1}P_{n-2} \quad \text{if } n \geqq 2, \tag{3.3}$$

it is better to state the recurrence in the following form:

$$P(0) := 1, \qquad P(1) := 1 + x_1 \tag{3.2$'$}$$
$$P(n) := P(n-1) + x_n P(n-1) + y_{n-1} P(n-2) \quad \text{if } n \geqq 2. \tag{3.3$'$}$$

The advantage of using subscripts to represent exponents is that the computer algebra system will not attempt to combine exponents, which would obscure the partitions being generated. Similarly, using x and y, two different variable names instead of a single variable q allows us to immediately distinguish between the blue parts and white parts being generated.

For our example, we find that

$P_0 = 1$
$P_1 = 1 + x_1$
$P_2 = 1 + x_1 + x_2 + x_2x_1 + y_1$
$P_3 = 1 + x_1 + x_2 + x_2x_1 + y_1 + x_3 + x_3x_1 + x_3x_2 + x_3x_2x_1 + x_3y_1 + y_2 + y_2x_1$
$\cdots\cdots$

The polynomials get large rather quickly, e.g. P_6 has over 150 terms, so we will not continue the list here, but upon carefully studying a list of, say, P_0 through P_6, it is quite reasonable to form the following conjecture (where blue parts are generated by the x's and white parts by the y's): The polynomials $P_n(q)$ generate partitions wherein

(1) the largest part is at most n,
(2) the largest white part is at most $n-1$,
(3) the number of blue parts plus twice the number of white parts is at most n,
(4) all parts are numerically distinct,
(5) if a white j appears as a part, then $j+1$ does not appear,
(6) if a white j appears as a part, then a white $j-1$ does not appear (although a blue $j-1$ may appear).

Next, we observe that for any n, the partitions represented by $P_n(q)$ are naturally divided into three disjoint classes, one for each term in (3.3$'$):

(1) those where the largest part is at most $n-1$ but the largest white part is at most $n-2$,
(2) those where the largest part is exactly n, and

(3) those where the largest part is a white $n-1$.

Then all we need to do is verify that these three disjoint classes match up with the recurrence expressed in (3.3′) to prove the conjecture.

Thus, we have already done most of the work towards proving the following combinatorial interpretation of (1.1):

Theorem 7 *Let $A(n)$ equal the number of two-color partitions of n into parts wherein*

(1) all parts are numerically distinct,
(2) if a white j appears, then $j+1$ does not appear (in either color),
(3) if a white j appears, then a white $j-1$ does not appear (although a blue $j-1$ may appear).

Let $B(n)$ equal the number partitions of n where each part is odd, and may be either blue or white. Then $A(n) = B(n)$ for all integers n.

PROOF. Upon taking the limit as $n \to \infty$ in $P_n(q)$, we see that the LHS of (1.1) is the generating function for partitions enumerated by $A(n)$.

The RHS of (1.1) is

$$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^4)_\infty}{(q;q^2)^2_\infty} = \prod_{j=0}^{\infty} \frac{(1-q^{4j+2})}{(1-q^{2j+1})^2} = \prod_{j=0}^{\infty} \left(1 + 2\sum_{m=0}^{\infty} q^{(2j+1)m}\right) \tag{5.1}$$

which is easily seen to be the generating function for partitions enumerated by $B(n)$. □

We note that Andrews [5, p. 121, Theorem 1] gives an alternate combinatorial interpretation of (1.1).

This last theorem is a combinatorial interpretation of (1.2).

Theorem 8 *Let $C(n)$ denote the number of two-color partitions of n into parts wherein*

(1) all parts are numerically distinct,
(2) if a white j appears as a part, then $j-1$ does not appear (in either color),
(3) if a white j appears as a part, then a white $j+1$ does not appear (although a blue $j+1$ may appear),
(4) no white 1's appear as parts.

Let $D(n)$ equal the number of ordinary (one color) partitions into nonmultiples of 4. Then $C(n) = D(n)$ for all integers n.

PROOF. The method of proof the same as that of the previous theorem. We note that in this case (following the notation from the proof of Theorem 3), we claim that polyomial $Q_n(q)$ generates partitions wherein

(1) the largest part is at most n,
(2) the number of blue parts plus twice the number of white parts is at most n,
(3) all parts are numerically distinct,
(4) if a white j appears as a part, then $j-1$ does not appear,
(5) if a white j appears as a part, then a white $j+1$ does not appear,
(6) no white 1's appear as parts.

Upon dividing the dividing the above partitions into three disjoint classes

(1) those with largest part less than n,
(2) those which contain a blue n, and
(3) those which contain a white n,

one simplify verifies that the above claim is satisfied by

$$Q(n) = Q(n-1) + x_n Q(n-1) + y_n Q(n-2).$$

Then, taking the limit as $n \to \infty$, the the LHS of (1.2) is seen to be the generating function of the partitions enumerated by $C(n)$. The RHS is clearly the generating function of the partitions enumerated by $D(n)$. $\square$

We also point out that a special case of Glaisher's Theorem [2, p. 6, Corollary 1.3] states that the number of partitions of n into nonmultiples of 4 equals the number of partitions of n where no part appears more than three times.

6 Conclusion

We conclude by pointing out that the two Lebesgue identities (1.1) and (1.2) are, in fact, special cases of the q-analog of Kummer's Theorem (see [8], [10], [1], and [2, p. 21, Corollary 2.7]). Thus, it is certainly possible that our combinatorial interpretations are special cases of a more general combinatorial phenomenon.

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