RRtools—a Maple package for aiding the discovery and proof of finite Rogers-Ramanujan type identities

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Abstract

The purpose of this paper is to introduce the `RRtools` and `recpf` Maple packages which were developed by the author to assist in the discovery and proof of finitizations of identities of the Rogers-Ramanujan type.

Key words: Rogers-Ramanujan Identities, q-Series, experimental mathematics

1 Introduction

The Rogers-Ramanujan identities, due to Rogers [1894], may be stated as follows:

The Rogers-Ramanujan Identities *If* $|q| < 1$, *then*

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.1}$$

and

$$\sum_{j=0}^{\infty} \frac{q^{j(j+1)}}{(q;q)_j} = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1.2}$$

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where

$$(a;q)_j = \prod_{m=0}^{j-1} (1 - aq^m),$$
$$(a;q)_\infty = \lim_{j\to\infty} (a;q)_j, \ \ \textit{and}$$
$$(a_1, a_2, \dots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \dots (a_r;q)_\infty.$$

There are many series-product identities which resemble the Rogers-Ramanujan identities in form, and are thus called "identities of the Rogers-Ramanujan type." The seminal papers on Rogers-Ramanujan type identities include Rogers [1894], Rogers [1917], Jackson [1928], Bailey [1947], Bailey [1949], and Slater [1952].

As documented by Berkovich and McCoy [1998], Rogers-Ramanujan type identities are essential to the solution of various models in statistical mechanics. In the language of the physicists, the left hand sides of (1.1) and (1.2) are called "fermionic" representations, and the right-hand sides are easily seen to be equivalent to what are called "bosonic" representations. For convenience, I will adopt this terminology, and use it throughout this paper.

Andrews [1981] showed that finitizations of (i.e. polynomial identities whose limiting cases are) Rogers-Ramanujan type identities are useful for determining relationships between various sets of identities via $q \to 1/q$ duality.

The `RRtools` package assists the user in finding finitizations of Rogers-Ramanujan type identities and duality relationships. In fact, with the use of this package, in [Sills, 2003], I was able to finitize all the identities in the list of Slater [1952].

There are at least three known categories of methods for finitizing Rogers-Ramanujan type identities:

(1) *Bailey's Lemma.* See Bailey [1949], Andrews [1986, Theorem 3.3], and Bressoud [1981].
(2) *Finitizations motivated by the models of statistical mechanics.* There are numerous papers in this area. Some representatives are Andrews [1981], Andrews et al. [1984], Berkovich and McCoy [1996], Berkovich and McCoy [1997], Berkovich et al. [1996], Berkovich et al. [1998a], Berkovich et al. [1998b], Schilling and Warnaar [1998], Warnaar [2001], Warnaar [2002].
(3) *Finitization via q-difference equations.* See Andrews [1986, Chapter 9], Santos [1991], Santos [2002], Santos and Sills [2002], Sills [2002], Sills [2003].

In some cases, two or more of the methods will produce identical finitizations of a given Rogers-Ramanujan type identity. However, in general, distinct finitizations will result. The `RRtools` package only deals with the method of q-difference equations.

After the mathematical preliminaries are reviewed in §2, I shall discuss finitization via q-difference equations in detail in §3.

A user's guide to the `RRtools` package is presented in §4, followed in §5 by a user's guide to the companion package `recpf`, which is a useful aid for proving identities conjectured with `RRtools`.

2 Mathematical Preliminaries

2.1 q-Binomial coëfficients

The *Gaussian polynomial* $\begin{bmatrix} A \\ B \end{bmatrix}_q$ may be defined as

$$\begin{bmatrix} A \\ B \end{bmatrix}_q := \begin{cases} \frac{(q;q)_A}{(q;q)_B (q;q)_{A-B}} & \text{if } 0 \leqq B \leqq A \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

The following properties of Gaussian polynomials are well-known [Andrews, 1976, pp. 35 ff.]. For $0 \leqq B \leqq A$ and $A > 0$, we have

$$\deg \left(\begin{bmatrix} A \\ B \end{bmatrix}_q \right) = B(A-B) \tag{2.2}$$

$$\lim_{q \to 1} \begin{bmatrix} A \\ B \end{bmatrix}_q = \binom{A}{B} \tag{2.3}$$

(Because of (2.3), Gaussian polynomials are also called *q-binomial coëfficients*.)

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{bmatrix} A \\ A-B \end{bmatrix}_q \tag{2.4}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{bmatrix} A-1 \\ B \end{bmatrix}_q + q^{A-B} \begin{bmatrix} A-1 \\ B-1 \end{bmatrix}_q \tag{2.5}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{bmatrix} A-1 \\ B-1 \end{bmatrix}_q + q^{B} \begin{bmatrix} A-1 \\ B \end{bmatrix}_q \tag{2.6}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}_{1/q} = q^{B(B-A)} \begin{bmatrix} A \\ B \end{bmatrix}_q \tag{2.7}$$

$$\lim_{n \to \infty} \begin{bmatrix} 2n+a \\ n+b \end{bmatrix}_q = \frac{1}{(q;q)_\infty}. \tag{2.8}$$

q-Binomial Theorem. *[Andrews et al., 1999, p. 488, Thm. 10.2.1] or [Andrews, 1976, p. 17, Thm. 2.1]. If $|t| < 1$ and $|q| < 1$,*

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} t^k = \frac{(at;q)_\infty}{(t;q)_\infty}. \tag{2.9}$$

We will make use of the following corollaries of (2.9):

$$\sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} t^k = (t;q)_j \tag{2.10}$$

$$\sum_{k=0}^{\infty} \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q t^k = \frac{1}{(t;q)_j} \tag{2.11}$$

$$\sum_{k=0}^{\infty} \frac{q^{rk^2+sk}}{(q^{2r};q^{2r})_k} = (-q^{r+s};q^{2r})_\infty \tag{2.12}$$

2.2 q-Trinomial coëfficients

2.2.1 Definitions

By expanding the Laurent polynomial $(1+x+x^{-1})^L$ and gathering like terms, we find

$$(1+x+x^{-1})^L = \sum_{j=-L}^{L} \binom{L}{j}_2 x^j \tag{2.13}$$

where

$$\binom{L}{A}_2 = \sum_{r=0} \frac{L!}{r!(r+A)!(L-2r-A)!} \tag{2.14}$$

$$= \sum_{r=0}^{L} (-1)^r \binom{L}{r} \binom{2L-2r}{L-A-r}. \tag{2.15}$$

The $\binom{L}{A}_2$ are called *trinomial coëfficients.*

The two representations (2.14) and (2.15) of $\binom{L}{A}_2$ give rise to different q-

analogs [Andrews and Baxter, 1987, p. 299, eqns. (2.7)–(2.12)]: [1]

$$\binom{L, B; q}{A}_2 := \sum_{r=0} \frac{q^{r(r+B)}(q;q)_L}{(q;q)_r (q;q)_{r+A} (q;q)_{L-2r-A}} = \sum_{r=0}^{L} q^{r(r+B)} \begin{bmatrix} L \\ r \end{bmatrix}_q \begin{bmatrix} L-r \\ r+A \end{bmatrix}_q \tag{2.16}$$

$$\mathrm{T}_0(L, A; q) := \sum_{r=0}^{L} (-1)^r \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q \tag{2.17}$$

$$\mathrm{T}_1(L, A; q) := \sum_{r=0}^{L} (-q)^r \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q \tag{2.18}$$

$$\tau_0(L, A; q) := \sum_{r=0}^{L} (-1)^r q^{Lr - \binom{r}{2}} \begin{bmatrix} L \\ r \end{bmatrix}_{q} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q \tag{2.19}$$

$$\mathrm{t}_0(L, A; q) := \sum_{r=0}^{L} (-1)^r q^{r^2} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q \tag{2.20}$$

$$\mathrm{t}_1(L, A; q) := \sum_{r=0}^{L} (-1)^j q^{r(r-1)} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_q \tag{2.21}$$

It is convenient to follow Andrews [1990] and define

$$\mathrm{U}(L, A; q) := \mathrm{T}_0(L, A; q) + \mathrm{T}_0(L, A+1; q). \tag{2.22}$$

Further, I will follow Sills [2003] and define

$$\mathrm{V}(L, A; q) := \mathrm{T}_1(L-1, A; q) + q^{L-A}\mathrm{T}_0(L-1, A-1; q). \tag{2.23}$$

2.2.2 Recurrences

The following Pascal triangle type recurrence may be deduced from (2.13):

$$\binom{L}{A}_2 = \binom{L-1}{A-1}_2 + \binom{L-1}{A}_2 + \binom{L-1}{A+1}_2. \tag{2.24}$$

We will require the following q-analogs of (2.24), which are due to Andrews and Baxter [1987, pp. 300-1, eqns. (2.16), (2.19), (2.25) (2.26), (2.28), and

[1] Note: Occasionally in the literature (e.g. [Andrews and Berkovich, 1998] or [Warnaar, 1999]), superficially different definitions of the T_0 and T_1 functions are used.

(2.29)]: For $L \geqq 1$,

$$\mathrm{T}_1(L,A;q) = \mathrm{T}_1(L-1,A;q) + q^{L+A}\mathrm{T}_0(L-1,A+1;q) + q^{L-A}\mathrm{T}_0(L-1,A-1;q) \tag{2.25}$$

$$\begin{aligned}\mathrm{T}_0(L,A;q) &= \mathrm{T}_0(L-1,A-1;q) + q^{L+A}\mathrm{T}_1(L-1,A;q)\\ &\quad + q^{2L+2A}\mathrm{T}_0(L-1,A+1;q)\end{aligned} \tag{2.26}$$

$$\begin{aligned}\binom{L,A-1;q}{A}_2 &= q^{L-1}\binom{L-1,A-1;q}{A}_2 + q^A\binom{L-1,A+1;q}{A+1}_2\\ &\quad + \binom{L-1,A-1;q}{A-1}_2\end{aligned} \tag{2.27}$$

$$\begin{aligned}\binom{L,A;q}{A}_2 &= q^{L-A}\binom{L-1,A-1;q}{A-1}_2 + q^{L-A-1}\binom{L-1,A-1;q}{A}_2\\ &\quad + \binom{L-1,A+1;q}{A+1}_2\end{aligned} \tag{2.28}$$

$$\begin{aligned}\binom{L,B;q}{A}_2 &= \binom{L-1,B;q}{A}_2 + q^{L-A-1+B}\binom{L-1,B;q}{A+1}_2\\ &\quad + q^{L-A}\binom{L-1,B-1;q}{A-1}_2\end{aligned} \tag{2.29}$$

$$\begin{aligned}\binom{L,B;q}{A}_2 &= \binom{L-1,B;q}{A}_2 + q^{L-A}\binom{L-1,B-2;q}{A-1}_2\\ &\quad + q^{L+B}\binom{L-1,B+1;q}{A+1}_2\end{aligned} \tag{2.30}$$

The following identities of Andrews and Baxter [1987, p. 301, eqns. (2.20) and (2.27 corrected)], which reduce to "$0 = 0$" in the $q = 1$ case are also useful:

$$\mathrm{T}_1(L,A;q) - q^{L-A}\mathrm{T}_0(L,A;q) - \mathrm{T}_1(L,A+1;q) + q^{L+A+1}\mathrm{T}_0(L,A+1;q) = 0, \tag{2.31}$$

$$\binom{L,A;q}{A}_2 + q^L\binom{L,A;q}{A+1}_2 - \binom{L,A+1;q}{A+1}_2 - q^{L-A}\binom{L,A-1;q}{A}_2 = 0. \tag{2.32}$$

Observe that (2.31) is equivalent to

$$\mathrm{V}(L+1,A+1;q) = \mathrm{V}(L+1,-A;q). \tag{2.33}$$

The following recurrences appear in Andrews [1990, p. 661, Lemmas 4.1 and

4.2]: For $L \geqq 1$,

$$\begin{aligned}\mathrm{U}(L,A;q) = (1+q^{2L-1})\mathrm{U}(L-1,A;q) + q^{L-A}\mathrm{T}_1(L-1,A-1;q)\\ +q^{L+A+1}\mathrm{T}_1(L-1,A+2;q).\end{aligned} \tag{2.34}$$

$$\begin{aligned}\mathrm{U}(L,A;q) = (1+q+q^{2L-1})\mathrm{U}(L-1,A;q) - q\mathrm{U}(L-2,A;q)\\ +q^{2L-2A}\mathrm{T}_0(L-2,A-2;q)\\ +q^{2L+2A+2}\mathrm{T}_0(L-2,A+3;q).\end{aligned} \tag{2.35}$$

An analogous recurrence for the "V" function [Sills, 2003, p. 7, eqn. (1.36)] is

$$\begin{aligned}\mathrm{V}(L,A;q) = (1+q^{2L-2})\mathrm{V}(L-1,A;q) + q^{L-A}\mathrm{T}_0(L-2,A-2;q)\\ +q^{L+A-1}\mathrm{T}_0(L-2,A+1;q).\end{aligned} \tag{2.36}$$

2.2.3 Identities

From (2.13), it is easy to deduce the symmetry relationship

$$\binom{L}{A}_2 = \binom{L}{-A}_2. \tag{2.37}$$

Two q-analogs of (2.37) are

$$\mathrm{T}_0(L,A;q) = \mathrm{T}_0(L,-A;q) \tag{2.38}$$

and

$$\mathrm{T}_1(L,A;q) = \mathrm{T}_1(L,-A;q). \tag{2.39}$$

The analogous relationship for the $\binom{L,B;q}{A}_2$ [Andrews and Baxter, 1987, p. 299, eqn. (2.15)] is

$$\binom{L,B;q}{-A}_2 = q^{A(A+B)}\binom{L,B+2A;q}{A}_2. \tag{2.40}$$

Other fundamental relations among the various q-trinomial coëfficients include the following (see Andrews and Baxter [1987, pp. 305–306]):

$$\binom{L,A;q}{A}_2 = \tau_0(L,A;q) \tag{2.41}$$

$$\mathrm{T}_0(L,A;q^{-1}) = q^{A^2-L^2}\mathrm{t}_0(L,A;q) = q^{A^2-L^2}\tau_0(L,A;q^2) \tag{2.42}$$

$$\mathrm{T}_1(L,A;q^{-1}) = q^{A^2-L^2}\mathrm{t}_1(L,A;q) \tag{2.43}$$

$$\tau_0(L,A;q^2) = \binom{L,A;q^2}{A}_2 = \mathrm{t}_0(L,A;q) \tag{2.44}$$

$$\binom{L,A-1;q^2}{A}_2 = q^{A-L}\mathrm{t}_1(L,A;q) \tag{2.45}$$

2.2.4 Asymptotics

The following asymptotic results for q-trinomial coëfficients are proved in, or are direct consequences of, Andrews and Baxter [1987, pp. 309–312]:

$$\lim_{L\to\infty}\binom{L,A;q}{A}_2 = \lim_{L\to\infty}\tau_0(L,A;q)=\frac{1}{(q;q)_\infty} \tag{2.46}$$

$$\lim_{L\to\infty}\binom{L,A-1;q}{A}_2 = \frac{1+q^A}{(q;q)_\infty} \tag{2.47}$$

$$\lim_{\substack{L\to\infty\\ L-A\text{ even}}}\mathrm{T}_0(L,A;q)=\frac{(-q;q^2)_\infty+(q,q^2)_\infty}{2(q^2;q^2)_\infty} \tag{2.48}$$

$$\lim_{\substack{L\to\infty\\ L-A\text{ odd}}}\mathrm{T}_0(L,A;q)=\frac{(-q;q^2)_\infty-(q;q^2)_\infty}{2(q^2;q^2)_\infty} \tag{2.49}$$

$$\lim_{L\to\infty}\mathrm{T}_1(L,A;q)=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \tag{2.50}$$

$$\lim_{L\to\infty}\mathrm{V}(L,A;q)=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \tag{2.51}$$

$$\lim_{L\to\infty}\mathrm{t}_0(L,A;q)=\frac{1}{(q^2;q^2)_\infty} \tag{2.52}$$

$$\lim_{L\to\infty}q^{-L}\mathrm{t}_1(L,A;q)=\frac{q^{-A}+q^A}{(q^2;q^2)_\infty} \tag{2.53}$$

$$\lim_{L\to\infty}\mathrm{U}(L,A;q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \tag{2.54}$$

2.3 Miscellaneous Results

The following well known identity is essential:

Jacobi's Triple Product Identity. *[Andrews, 1976, p. 21, Theorem 2.8] or [Ismail, 1977]. For $z\neq 0$ and $|q|<1$,*

$$\begin{aligned}\sum_{j=-\infty}^{\infty} z^j q^{j^2} &= \prod_{j=1}^{\infty}(1+zq^{2j-1})(1+z^{-1}q^{2j-1})(1-q^{2j}) \\ &= (-zq,-z^{-1}q,q^2;q^2)_\infty\end{aligned} \tag{2.55}$$

The following two results can be used to simplify certain sums of two instances of Jacobi's triple product identity:

Quintuple Product Identity. *[Watson, 1929]*

$$\prod_{j=1}^{\infty}(1+z^{-1}q^j)(1+zq^{j-1})(1-z^{-2}q^{2j-1})(1-z^2q^{2j-1})(1-q^j) \tag{2.56}$$
$$=\prod_{j=1}^{\infty}(1-z^3q^{3j-2})(1-z^{-3}q^{3j-1})(1-q^{3j})+z\prod_{j=1}^{\infty}(1-z^{-3}q^{3j-2})(1-z^3q^{3j-1})(1-q^{3j})$$

or, in abbreviated notation,

$$(z^3q,z^{-3}q^2,q^3;q^3)_\infty+z(z^{-3}q,z^3q^2,q^3;q^3)_\infty=(-z^{-1}q,-z,q;q)_\infty(z^{-2}q,z^2q;q^2)_\infty.$$

Next, an identity of Bailey [1951, p. 220, eqn. (4.1)]:

$$\prod_{j=1}^{\infty}(1+z^2q^{4j-3})(1+z^{-2}q^{4j-1})(1-q^{4j})+z\prod_{j=1}^{\infty}(1+z^2q^{4j-1})(1+z^{-2}q^{4j-3})(1-q^{4j}) \tag{2.57}$$
$$=\prod_{j=1}^{\infty}(1+zq^{j-1})(1+z^{-1}q^j)(1-q^j)$$

or, in abbreviated notation,

$$(-z^2q,-z^{-2}q^3,q^4;q^4)_\infty+z(-z^2q^3,-z^{-2}q,q^4;q^4)_\infty=(-z,-z^{-1}q,q;q)_\infty.$$

We will also require the following result:

Abel's Lemma. *[Whittaker and Watson, 1958, p. 57] or [Andrews, 1971, p. 190]. If $\lim_{n\to\infty}a_n=L$, then*

$$\lim_{t\to 1^-}(1-t)\sum_{n=0}^{\infty}a_nt^n=L. \tag{2.58}$$

3 Finitization and Duality for Rogers-Ramanujan Type Identities

3.1 Finitization by the Method of q-Difference Equations

We now turn our attention to a method for discovering finite analogs of Rogers-Ramanujan type identities, i.e. polynomial identities which converge to a given identity of the Rogers-Ramanujan type. As there may be many distinct polynomial identities which converge to a given series-product identity, there may be many different ways to finitize a given series-product identity. For a discussion of other methods which in general lead to finitizations different from those we will discuss here, see the discussion in §0 of Sills [2003]. Here I will focus on one particular method of finitization which will be referred to as the "method of q-difference equations." I have automated much of the process in the `RRtools` package.

The method of q-difference equations was first discussed by Andrews [1986, sec. 9.2, p. 88]. We begin with an identity of the Rogers-Ramanujan type

$$\phi(q) = \Pi(q)$$

where $\phi(q)$ is the series and $\Pi(q)$ is an infinite product or sum of several infinite products. We consider a two variable generalization $f(q,t)$ which satisfies the following three conditions:

Conditions 1

(1) $f(q,t) = \sum_{n=0}^{\infty} P_n(q) t^n$ *where the* $P_n(q)$ *are polynomials,*
(2) $\phi(q) = \lim_{t\to 1^-}(1-t)f(q,t) = \lim_{n\to\infty} P_n(q) = \Pi(q)$*, and*
(3) $f(q,t)$ *satisfies a nonhomogeneous q-difference equation of the form*

$$f(q,t) = R_1(q,t) + R_2(q,t) f(q, tq^k)$$

where $R_i(q,t)$ *are rational functions of* q *and* t *for* $i = 1,2$ *and* $k \in \mathbb{Z}_+$.

Sills [2003, p. 15, Theorem 2.2] has shown that if $\phi(q)$ is written in the form

$$\sum_{j=0}^{\infty} \frac{(-1)^{aj} q^{bj^2+cj} \prod_{i=1}^{r} (d_i q^{e_i}; q^{k_i})_{j+l_i}}{(q^m;q^m)_j \prod_{i=1}^{s} (\delta_i q^{\epsilon_i}; q^{\kappa_i})_{j+\lambda_i}},$$

where $a = 0$ or 1; $b, m \in \mathbb{Z}_+$; $c \in \mathbb{Z}$;
$d_i = \pm 1$; $e_i, k_i \in \mathbb{Z}_+$, $l_i \in \mathbb{Z}$ for $1 \leqq i \leqq r$;
$\delta_i = \pm 1$; $\epsilon_i, \kappa_i \in \mathbb{Z}_+$; $\lambda_i \in \mathbb{Z}$ for $1 \leqq i \leqq s$; then

$$f(q,t) = \sum_{j=0}^{\infty} \frac{(-1)^{aj} t^{2bj/g} q^{bj^2+cj} \prod_{i=1}^{r} (d_i t^{k_i/g} q^{e_i}; q^{k_i})_{j+l_i}}{(t;q^m)_{j+1} \prod_{i=1}^{s} (\delta_i t^{\kappa_i/g} q^{\epsilon_i}; q^{\kappa_i})_{j+\lambda_i}},$$

where $g = gcd(m, k_1, k_2, \ldots, k_r, \kappa_1, \kappa_2, \ldots, \kappa_s)$ is a two variable generalization of $\phi(q)$ which satisfies Conditions 1.

The nonhomogeneous q-difference equation can be used to find a recurrence which the $P_n(q)$ satisfy, and thus a list of $P_0(q), P_1(q), \ldots, P_N(q)$ can be produced for any N.

The fermionic representation of the finitization is obtained by expanding the rising q-factorials which appear in $f(q,t)$ using (2.10) and (2.11), changing variables so that the resulting power of t is n, so that $P_n(q)$ can be seen as the coëfficient of t^n.

Obtaining the bosonic representation for $P_n(q)$ is more difficult, and requires conjecturing the correct form. As we shall see, the `RRtools` Maple package contains a number of tools to aid the user in making an appropriate conjecture.

After the proposed polynomial identity is conjectured, it can be proved showing that the bosonic representation satisfies the same recurrence and initial conditions as the fermionic representation.

3.2 An Example Done "by Hand"

To serve as a prototypical example, we will examine the finitization process on an identity from the list of Slater [1952, p. 153, eqn. (7)], an identity due to Euler:

$$\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q^2;q^2)_j} = \prod_{j=1}^{\infty} (1+q^{2j}) \tag{3.1}$$

The required two variable generalization of the LHS of (3.1) is

$$f(q,t) = \sum_{j=0}^{\infty} \frac{t^j q^{j^2+j}}{(1-t)(tq^2;q^2)_j}.$$

Next, we produce the nonhomogeneous q-difference equation.

$$\begin{aligned}
f(q,t) &= \sum_{j=0}^{\infty} \frac{t^j q^{j^2+j}}{(t;q^2)_{j+1}} \\
&= \frac{1}{1-t} + \sum_{j=1}^{\infty} \frac{t^j q^{j^2+j}}{(t;q^2)_{j+1}} \\
&= \frac{1}{1-t} + \sum_{j=0}^{\infty} \frac{t^{j+1} q^{(j+1)^2+(j+1)}}{(t;q^2)_{j+2}} \\
&= \frac{1}{1-t} + \frac{tq^2}{1-t} \sum_{j=0}^{\infty} \frac{(tq^2)^j q^{j^2+j}}{(tq^2;q^2)_{j+1}} \\
&= \frac{1}{1-t} + \frac{tq^2}{1-t} f(q,tq^2)
\end{aligned}$$

Thus, a non-homogeneous q-difference equation satisfied by $f(q,t)$ is

$$f(q,t) = \frac{1}{1-t} + \frac{tq^2}{1-t} f(q,tq^2). \tag{3.2}$$

Next, we find the sequence of polynomials $\{P_n(q)\}_{n=0}^{\infty}$ as follows:

Clearing denominators in (3.2) gives

$$(1-t)f(q,t) = 1 + tq^2 f(q,tq^2),$$

which is equivalent to

$$f(q,t) = 1 + tf(q,t) + tq^2 f(q,tq^2).$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(q)t^n &= 1 + t\sum_{n=0}^{\infty} P_n(q)t^n + tq^2 \sum_{n=0}^{\infty} P_n(q)(tq^2)^n \\
&= 1 + \sum_{n=0}^{\infty} P_n(q)t^{n+1} + \sum_{n=0}^{\infty} P_n(q)t^{n+1}q^{2n+2} \\
&= 1 + \sum_{n=1}^{\infty} P_{n-1}(q)t^n + \sum_{n=1}^{\infty} q^{2n}P_{n-1}(q)t^n \\
&= 1 + \sum_{n=1}^{\infty} (1+q^{2n})P_{n-1}(q)t^n.
\end{aligned}$$

We can read off from the last line that the polynomial sequence $\{P_n(q)\}_{n=0}^{\infty}$ satisfies the following recurrence relation:

$$P_0(q) = 1 \tag{3.3}$$
$$P_n(q) = (1+q^{2n})P_{n-1}, \text{ if } n \geqq 1. \tag{3.4}$$

Note that for this example, since a first order recurrence was obtained, $P_n(q)$ is expressible as a finite product, and thus in some sense, the problem is done. However, the overwhelming majority of the identities from Slater's list yield finitizations whose minimal recurrence order is greater than one, and thus *not* expressible as a finite product. In such cases, we must work harder to find a representation for $P_n(q)$ which can be seen to converge in a direct fashion to the RHS of the original identity. Thus we continue the demonstration:

Now that a recurrence for the $P_n(q)$ is known, a finite list $\{P_n(q)\}_{n=0}^{N}$ can be produced:

$$\begin{aligned}
P_0(q) &= 1 \\
P_1(q) &= q^2 + 1 \\
P_2(q) &= q^6 + q^4 + q^2 + 1 \\
P_3(q) &= q^{12} + q^{10} + q^8 + 2q^6 + q^4 + q^2 + 1 \\
P_4(q) &= q^{20} + q^{18} + q^{16} + 2q^{14} + 2q^{12} + 2q^{10} + 2q^8 + 2q^6 + q^4 + q^2 + 1
\end{aligned}$$

Notice that the degree of $P_n(q)$ appears to be $n(n+1)$. Being familiar with Gaussian polynomials, we recall that the degree of $\left[{2n+1 \atop n+1}\right]_q$ is also $n(n+1)$ (by (2.2)), and wonder if this Gaussian polynomial might play a fundamental rôle in the bosonic representation of $P_n(q)$. Also, since

$$\Pi(q) = \frac{(q, q^3, q^4; q^4)_\infty}{(q;q)_\infty}$$

(an instance of Jacobi's triple product identity multiplied by $1/(q;q)_\infty$) and

$$\lim_{n\to\infty} \begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q = \frac{1}{(q;q)_\infty} \quad \text{(by (2.8))},$$

we have further evidence in favor of the Gaussian polynomial $\left[{2n+1 \atop n+1}\right]_q$ playing a central rôle. Using the method of successive approximations by Gaussian polynomials discussed by Andrews and Baxter [1990], one can conjecture that, at least for small n, it is true that

$$P_n(q) = \begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q - q\begin{bmatrix} 2n+1 \\ n+2 \end{bmatrix}_q - q^3\begin{bmatrix} 2n+1 \\ n+3 \end{bmatrix}_q + q^6\begin{bmatrix} 2n+1 \\ n+4 \end{bmatrix}_q + q^{10}\begin{bmatrix} 2n+1 \\ n+5 \end{bmatrix}_q - \dots,$$

which is a good start, but the bosonic representation must be a *bi*lateral series. Thus we employ (2.4) to rewrite the above as

$$P_n(q) = \begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q - q\begin{bmatrix} 2n+1 \\ n-1 \end{bmatrix}_q - q^3\begin{bmatrix} 2n+1 \\ n+3 \end{bmatrix}_q + q^6\begin{bmatrix} 2n+1 \\ n-3 \end{bmatrix}_q + q^{10}\begin{bmatrix} 2n+1 \\ n+5 \end{bmatrix}_q - \dots,$$

which is equivalent to

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n+2j+1 \end{bmatrix}_q, \tag{3.5}$$

which is in the desired (bosonic) form.

Obtaining the fermionic representation for $P_n(q)$ simply requires expanding the q-factorials by (2.10) or (2.11) as appropriate:

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(q)t^n &= f(q,t) \\
&= \sum_{j=0}^{\infty} \frac{t^j q^{j^2+j}}{(1-t)(tq^2;q^2)_j} \\
&= \sum_{j=0}^{\infty} t^j q^{j^2+j} \sum_{k=0}^{\infty} \begin{bmatrix} j+k \\ k \end{bmatrix}_{q^2} t^k \text{ by (2.11)}
\end{aligned}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} t^{j+k} q^{j^2+j} \begin{bmatrix} j+k \\ j \end{bmatrix}_{q^2} \text{ by (2.4)}$$
$$= \sum_{n=0}^{\infty} t^n \sum_{j=0}^{\infty} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \text{ (by taking } n = j+k\text{)}$$

By comparing coëfficients of t^n in the extremes, we find

$$P_n(q) = \sum_{j=0}^{\infty} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}. \tag{3.6}$$

Combining (3.6) and (3.5), we obtain the conjectured polynomial identity

$$\sum_{j=0}^{\infty} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n+2j+1 \end{bmatrix}_{q}. \tag{3.7}$$

We now need to prove that identity (3.7) is valid.

Remark. Equation (3.7) may be proved in a completely automated fashion by the q-Zeilberger algorithm. I include a proof by recurrence here as a simple example to serve as a prototype for the type of proofs assisted by the `recpf` package, to be described in section 5. In fact, all the identities in §3 of Sills [2003] are *theoretically* provable by the q-Zeilberger algorithm [Zeilberger, 1991] or its multisum generalization [Wilf and Zeilberger, 1992]. However, in practice, the `qZeil` [Paule and Riese, 1997] and `qMultiSum` [Riese, 2003] Mathematica packages, which are truly state-of-the-art, nonetheless encounter run time and complexity difficulties when attempting to deal with many of these identities [Riese, 2002].

PROOF OF (3.7). The fermionic representation (i.e. lefthand side) of (3.7) is known to satisfy the recurrence (3.4) and the initial condition (3.3) by the way it was constructed. Thus, the proof of (3.7) will be complete upon showing that the proposed bosonic representation (i.e. righthand side) of (3.7) satisfies (3.4) and (3.3).

First, it is trivial to check that the $n = 0$ case of (3.7) holds; it reduces to $1 = 1$.

Next, we want to show that the RHS of (3.7) satisfies (3.4), or equivalently that it satisfies

$$P_n(q) - (1 + q^{2n})P_{n-1}(q) = 0. \tag{3.8}$$

So substitute the RHS of (3.7) for $P_n(q)$ in the LHS of (3.8):

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n+2j+1 \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q$$

Next, apply (2.6) to the first term to obtain

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n \\ n+2j \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+n+1} \begin{bmatrix} 2n \\ n+2j+1 \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q$$

Then apply (2.5) to the first term to obtain

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+n+1} \begin{bmatrix} 2n \\ n+2j+1 \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q$$

whereupon the first and fourth terms cancel leaving

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+n+1} \begin{bmatrix} 2n \\ n+2j+1 \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q$$

Next, we apply (2.5) to the second term to get

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+n+1} \begin{bmatrix} 2n-1 \\ n+2j+1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j+2n} \begin{bmatrix} 2n-1 \\ n+2j \end{bmatrix}_q$$

and so the third and fourth terms cancel leaving

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+3j+n+1} \begin{bmatrix} 2n-1 \\ n+2j+1 \end{bmatrix}_q$$

By replacing j with $j-1$ in the second term, we now have

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q + \sum_{j=-\infty}^{\infty} (-1)^{j-1} q^{2(j-1)^2+3(j-1)+n+1} \begin{bmatrix} 2n-1 \\ n+2(j-1)+1 \end{bmatrix}_q$$

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q - \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2-j+n} \begin{bmatrix} 2n-1 \\ n+2j-1 \end{bmatrix}_q = 0$$

and thus the recurrence (3.8) is satisfied by the RHS of (3.7) and the identity (3.7) is proved. $\square$

To see that (3.7) is indeed a finitization of (3.1), we carry out the following calculations:

$$\begin{aligned} &\lim_{n\to\infty} \sum_{j=0}^{\infty} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \\ &= \lim_{n\to\infty} \sum_{j=0}^{\infty} q^{j^2+j} \frac{(q^2;q^2)_n}{(q^2;q^2)_j (q^2;q^2)_{n-j}} \\ &= \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q^2;q^2)_j}, \end{aligned}$$

and so the LHS of (3.7) converges to the LHS of (3.1).

$$\begin{aligned} &\lim_{n\to\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \begin{bmatrix} 2n+1 \\ n+2j+1 \end{bmatrix}_q \\ &= \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} \quad \text{(by (2.8))} \\ &= \frac{1}{(q;q)_\infty} \cdot (q, q^3, q^4; q^4)_\infty \quad \text{(by Jacobi's Triple Product Identity (2.55))} \\ &= \prod_{j=1}^{\infty} (1+q^{2j}), \end{aligned}$$

and so the RHS of (3.7) converges to the RHS of (3.1), and thus identity (3.1) may be viewed as a corollary of identity (3.7).

3.3 Duality

Andrews [1981] demonstrated a type of duality relationship that exists among a few sets of Rogers-Ramanujan type identities. The *(reciprocal) dual* of a polynomial

$$a_n q^n + a_{n-1}q^{n-1} + a_{n-2}q^{n-2} + \cdots + a_1 q + a_0$$

(where $a_n \neq 0$) is

$$a_0 q^n + a_1 q^{n-1} + a_2 q^{n-2} + \cdots + a_{n-1} q + a_n.$$

Equivalently, the reciprocal of $P(q)$ is

$$q^{\deg(P(q))} P(q^{-1}). \tag{3.9}$$

If $q^{\deg(P(q))} P(q^{-1}) = P(q)$, the associated identity is called *self-dual.* Let us work through an example of calculating the dual of an identity: Consider, say, identity 10 of Slater [1952], which is

$$\sum_{j=0} \frac{(-1;q)_{2j} q^{j^2}}{(q^2;q^2)_j (q^2;q^4)_j} = \prod_{j=1} (1+q^{2j-1})(1+q^j) \tag{3.10}$$

A finite form of (3.10) is

$$\begin{aligned} &\sum_{i=0}\sum_{j=0}\sum_{k=0} q^{j^2+i^2-i+k} \begin{bmatrix} j \\ i \end{bmatrix}_{q^2} \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-i-k \\ j \end{bmatrix}_{q^2} \\ &\quad = \sum_{j=-\infty}^{\infty} q^{2j^2+j} \Big[\mathrm{T}_0(n,2j;q) + \mathrm{T}_0(n-1,2j;q) \Big]. \end{aligned} \tag{3.11}$$

(See [Sills, 2003, Identity 3.10].) Notice that when $j = k = 0$ in the LHS sum, $\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]_{q^2}$ must be interpreted as 1, even though (2.1) would indicate $\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]_{q^2} = 0$. Such flexibility regarding initial conditions is often required when dealing with Gaussian polynomials.

Remark. One referee pointed out that the LHS of (3.11) can be simplified to a double sum using a special case of the Andrews-Askey formula [Gasper and Rahman, 1990, p. 22, ex (1.8), $b = q^{-2n}$, $a = q^{j-n}$]. Thus we have

$$\sum_{k=0} q^k \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-k \\ j \end{bmatrix}_{q^2} = \begin{bmatrix} n+j \\ 2j \end{bmatrix}_q.$$

This points to one of the inherant limitations of `RRtools`. Since the series side of Identity 10 of Slater [1952] contains three rising q-factorials, `RRtools` will automatically find a three-fold sum. For a number of identities in Sills [2003], I noticed such a simplification and worked it out by hand. For others, such as (3.11), I missed the simplification. For still others, a reduction from a three-fold to a two-fold sum was possible, but not desirable, as it obscured the fact that the limit as $n \to \infty$ was, in fact, the original series.

An examination of the first few cases $n = 0, 1, 2$, etc., convinces one that the degree of the polynomial is n^2. Thus, by (3.11) and (3.9), the dual polynomials of (3.11) can be represented by either of the two forms

$$\begin{aligned}
&\sum_{i=0}\sum_{j=0}\sum_{k=0} q^{n^2-(j^2+i^2-i+k)} \begin{bmatrix} j \\ i \end{bmatrix}_{1/q^2} \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_{1/q^2} \begin{bmatrix} n-i-k \\ j \end{bmatrix}_{1/q^2} \\
&= \sum_{j=-\infty}^{\infty} q^{n^2-(2j^2+j)} \Big[\mathrm{T}_0(n, 2j; 1/q) + \mathrm{T}_0(n-1, 2j; 1/q) \Big]. \qquad (3.12)
\end{aligned}$$

Applying (2.42) and (2.41) on the righthand side and (2.7) on the left hand side, we obtain

$$\begin{aligned}
&\sum_{h=0}\sum_{i=0}\sum_{k=0} q^{k+i+2i(h+i+k)+(h+k)^2} \begin{bmatrix} n-i-h-k \\ i \end{bmatrix}_{q^2} \begin{bmatrix} n-i-h-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-i-k \\ h \end{bmatrix}_{q^2} \\
&\qquad = \sum_{j=-\infty}^{\infty} q^{2j^2-j} \binom{n, 2j; q^2}{2j}_2 + q^{2j^2-j+2n-1} \binom{n-1, 2j; q^2}{2j}_2 .
\end{aligned}$$

We can suppose that $|q| < 1$ and let $n \to \infty$ in the preceding equation to obtain a Rogers-Ramanujan type identity. First, we consider the lefthand side:

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{h,i,k=0} q^{k+i+2i(h+i+k)+(h+k)^2} \begin{bmatrix} n-i-h-k \\ i \end{bmatrix}_{q^2} \begin{bmatrix} n-i-h-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-i-k \\ h \end{bmatrix}_{q^2} \\
&= \lim_{n\to\infty} \sum_{h,i,k=0} \frac{q^{k+i+2i(h+i+k)+(h+k)^2} (q^2;q^2)_{n-i-j-k} (q^2;q^2)_{n-i-h-1} (q^2;q^2)_{n-i-k}}{(q^2;q^2)_i (q^2;q^2)_{n-2i-h-k} (q^2;q^2)_k (q^2;q^2)_{n-i-k-h-1} (q^2;q^2)_h (q^2;q^2)_{n-i-k-h}} \\
&\qquad = \sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{k+i+2i(h+i+k)+(h+k)^2}}{(q^2;q^2)_h (q^2;q^2)_i (q^2;q^2)_k}
\end{aligned}$$

Next, we consider the righthand side:

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{j=-\infty}^{\infty} q^{2j^2-j} \binom{n, 2j; q^2}{2j}_2 + q^{2j^2-j+2n-1} \binom{n-1, 2j; q^2}{2j}_2 \\
&= \frac{1}{(q^2;q^2)_\infty} \left[\sum_{j=-\infty}^{\infty} q^{2j^2-j} + 0 \right] \qquad \text{(by (2.46) and since } |q| < 1) \\
&= \frac{1}{(q^2;q^2)_\infty} (-q, -q^3, q^4; q^4)_\infty \qquad \text{(by (2.55))} \\
&= (-q;q)_\infty \\
&= \prod_{j=1}^{\infty} (1+q^j)
\end{aligned}$$

Thus, for $|q| < 1$,

$$\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{k+i+2i(h+i+k)+(h+k)^2}}{(q^2;q^2)_h(q^2;q^2)_i(q^2;q^2)_k} = \prod_{j=1}^{\infty}(1+q^j). \tag{3.13}$$

Note that the triple sum in the preceeding equation can be simplified:

$$\begin{aligned}
&\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{k+i+2i(h+i+k)+(h+k)^2}}{(q^2;q^2)_h(q^2;q^2)_i(q^2;q^2)_k}\\
&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{k+i+i^2+(i+k)^2}}{(q^2;q^2)_i(q^2;q^2)_k}\sum_{h=0}^{\infty}\frac{q^{h^2+(2i+2k)h}}{(q^2;q^2)_h}\\
&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{k+i+i^2+(i+k)^2}}{(q^2;q^2)_i(q^2;q^2)_k}(-q^{2i+2k+1};q^2)_\infty \quad \text{(by (2.12))}\\
&=(-q;q^2)_\infty\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{k^2+2ik+2i^2+k+i}}{(q^2;q^2)_i(q^2;q^2)_k(-q;q^2)_{i+k}}\\
&=(-q;q^2)_\infty\sum_{i=0}^{\infty}\sum_{K=i}^{\infty}\frac{q^{K^2+K+i^2}}{(q^2;q^2)_i(q^2;q^2)_{K-i}(-q;q^2)_K} \quad \text{(by taking } K=k+i\text{)}\\
&=(-q;q^2)_\infty\sum_{K=0}^{\infty}\frac{q^{K^2+K}}{(-q;q^2)_K}\sum_{i=0}^{K}\frac{q^{i^2}}{(q^2;q^2)_i(q^2;q^2)_{K-i}}\\
&=(-q;q^2)_\infty\sum_{K=0}^{\infty}\frac{q^{K^2+K}}{(-q;q^2)_K(q^2;q^2)_K}\sum_{i=0}^{K}\frac{q^{i^2}(q^2;q^2)_K}{(q^2;q^2)_i(q^2;q^2)_{K-i}}\\
&=(-q;q^2)_\infty\sum_{K=0}^{\infty}\frac{q^{K^2+K}}{(-q;q^2)_K(q^2;q^2)_K}\sum_{i=0}^{K}q^{i^2}\begin{bmatrix}K\\ i\end{bmatrix}_{q^2}\\
&=(-q;q^2)_\infty\sum_{K=0}^{\infty}\frac{q^{K^2+K}(-q;q^2)_K}{(-q;q^2)_K(q^2;q^2)_K} \quad \text{(by (2.10))}\\
&=(-q;q^2)_\infty\sum_{K=0}^{\infty}\frac{q^{K^2+K}}{(q^2;q^2)_K}.
\end{aligned}$$

Thus, (3.13) is equivalent to

$$(-q;q^2)_\infty\sum_{j=0}^{\infty}\frac{q^{j(j+1)}}{(q^2;q^2)_j} = \prod_{j=1}^{\infty}(1+q^j),$$

or, after dividing through by $(-q;q^2)_\infty$,

$$\sum_{j=0}^{\infty}\frac{q^{j(j+1)}}{(q^2;q^2)_j} = \prod_{j=1}^{\infty}(1+q^{2j}),$$

which is identity (3.1). I will not go so far as to say that “Identities (3.1) and (3.10) are dual” since it was necessary to take the limit as $n \to \infty$ and to divide out an infinite product in order to obtain (3.1) from the dual of (3.10). In fact, it is not hard to show that identity (3.1) is *self*-dual under our finitization.

In some cases, the sequence of polynomials $\{P_n(q)\}_{n=0}^{\infty}$ does not converge, but the subsequence $\{P_{2m}(q)\}_{m=0}^{\infty}$ converges to one series, and the subsequence $\{P_{2m+1}(q)\}_{m=0}^{\infty}$ converges to a different series. One immediate clue that this may be occuring is when the formula for the degree of the polynomial varies with the parity of n. For example, with the finite First Rogers-Ramanujan Identity (4.2), the degree of the polynomial is $n^2/4$ if n is even, and $(n^2-1)/4$ if n is odd. In cases such as this, we consider the dual of (4.2) to be a pair of identities. The appropriate calculation shows that the dual of (4.2-even) is identity 79 on the list of Slater [1952], while the dual of (4.2-odd) is another identity [Slater, 1952, eqn. (99)].

4 The `RRtools` Maple package

The `RRtools` that I developed to aid my research for Sills [2002] and Sills [2003] contains a variety of tools which may be of interest to others. This user's guide assumes basic familiarity with the Maple computer algebra system. I developed the package using Maple V release 4, and later tested it (making necessary modifications) for use with Maple 6 and Maple 7. I believe that it is fully functional on at least these three versions of Maple. Please send me an e-mail if (when) you find bugs.

User input is indicated in `non-proportional "typewriter" style font` and preceded by a Maple prompt `>`, which of course is *not* to be typed. Maple's responses will be indicated directly below in standard mathematical italics.

4.1 Setup and Initialization

The `RRtools` package may be downloaded free of charge from my web site `http://www.math.rutgers.edu/~asills`. Copy the file `RRtools1` into the directory in which you intend to initiate your Maple session. Begin your Maple session as usual. Type

```
> read(RRtools1);
```

Maple responds with a welcome message which includes a list of the procedures in the package.

4.2 Basic q-Functions

The Gaussian polynomial $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]_{q^r}$ is input as `gp(A,B,r)`. `RRtools` contains the analogous procedures for the q-trinomial coëfficients `T0, T1, tau0, t0, t1` and the related functions `U` and `V`. For example,

```
> gp(5,2,3);
```

returns

$$q^{18} + q^{15} + 2q^{12} + 2q^9 + 2q^6 + q^3 + 1$$

and

```
> U(2,1,1);
```

returns

$$q^3 + q + 1.$$

Also supported is the finite rising q-factorial $(a;q)_n$ which is entered in the form `qfact(a,q,n)`, for example,

```
> qfact(a,q^2,3);
```

returns

$$(1-a)(1-aq^2)(1-aq^4).$$

4.3 q-Difference Equations and Recurrences

For a given Rogers-Ramanujan Type series $\phi(q)$, the two variable generalization $f(q,t)$ which satisfies Conditions 1 is produced using the `twovargen` procedure. The input format is `twovargen[`*summand*, *lowerlim*`]` where *summand* may contain $(-1)^j$, rising q-factorials in the numerator and denominator and *must* contain q raised to a quadratic power in j in the numerator and exactly one specially designated q-factorial in the denominator to which the extra $(1-t)$ factor is attached. The argument *lowerlim* is either 0 or 1, depending on whether the series is of the form $\sum_{j=0}^{\infty}$ or $1+\sum_{j=1}^{\infty}$ respectively.

For example, to find the $f(q,t)$ associated with identity 60 from the list of Slater [1952] which is

$$\phi(q) = \sum_{j=0}^{\infty} \frac{q^{j(j+1)}}{(q;q^2)_{j+1}(q;q)_j},$$

type

```
> twovargen( q^(j*(j+1)) / (qfac(q,q^2,j+1) * spqfac(q,q,j)),0);
```

and Maple returns

$$\sum_{j=0}^{\infty} \frac{t^{(2j)} q^{(j^2+j)}}{(t^2q, q^2)_{j+1}(t,q)_{j+1}}.$$

Due to a limitation in the Maple output procedure, the q-factorial $(a;q)_j$ appears instead as $(a,q)_j$.

Note the difference between "qfact" and "qfac": the former returns the rising q-factorial as a finite product (explicitly if the third argument is an integer, and in Maple product notation if the third argument is a variable), while the latter is an inert form to be used in the input argument. For example,

```
> qfact(a,q,3);
```

$$(1-a)(1-aq)(1-aq^2)$$

```
> qfac(a,q,3);
```

$$\mathrm{qfac}(a,q,3)$$

```
> qfact(a,q,j);
```

$$\prod_{m=0}^{j-1}(1-aq^m)$$

```
> qfac(a,q,j);
```

$$\mathrm{qfac}(a,q,j)$$

Also note that "spqfac" (i.e. "special q-factorial") is distinguished from "qfac" because it is the q-factorial to which the extra $(1-t)$ factor is attached.

To find the $f(q,t)$ associated with

$$1+\sum_{j=0}^{\infty} \frac{(q^2;q^2)_{j-1}q^{j^2}}{(q;q^2)_j(q;q)_{j-1}(q;q)_j},$$

the LHS of identity 58 from the list of Slater [1952], type

```
> twovargen( q^(j^2) * qfac(q^2,q^2,j-1) / ( qfac(q,q^2,j) *

 qfac(q,q,j-1) * spqfac(q,q,j)), 1);
```

If $\phi(q)$ contains a q-factorial where the coëfficient of j in the index is not 1, as in $(q^2;q^2)_{2j+1}$ in [Slater, 1952, p. 166, eqn. (125)],

$$\sum_{n=0}^{\infty} \frac{(q^3;q^6)_j q^{2j(j+2)}}{(q^2;q^2)_{2j+1}(q;q^2)_j},$$

the user must rewrite $(q^2;q^2)_{2j+1}$ as $(q^2;q^4)_{j+1}(q^4;q^4)_j$:

```
> twovargen( qfac(q^3,q^6,j) * q^(2*j*(j+2)) /

(qfac(q^2,q^4,j+1) * qfac(q,q^2,j) * spqfac(q^4,q^4,j)), 0);
```

The `funcrecur` procedure returns the nonhomogeneous q-difference equation satisfied by $f(q,t)$. The input is the same as in `twovargen`, plus an additional parameter at the end to be used as a label (such as the identity number from Slater's list).

For example, to find the q-difference equation associated with the first Rogers-Ramanujan Identity (1.1) type

```
> funcrecur(q^(j^2)/spqfac(q,q,j), 0, 18)
```

and Maple returns

$$\begin{gathered} f_{18}(t) = \frac{1}{1-t} + \left(\frac{t^2 q}{1-t}\right) f_{18}(tq) \\ (1-t)f_{18}(t) = (1) + (t^2 q) f_{18}(tq). \end{gathered}$$

Next, we want to know the recurrence and initial conditions satisfied by the polynomials $P_n(q)$, for which $f(q,t)$ is a generating function:

```
> polyrecur(q^(j^2)/spqfac(q,q,j),0);
```

$$P_n = P_{n-1} + P_{n-2} q^{(n-1)}$$

```
> initialconds(q^(j^2)/spqfac(q,q,j),0);
```

$$\begin{gathered} P_0 = 1 \\ P_1 = 1 \end{gathered}$$

The n-shift operator N acts on $F(n,j)$ by

$$NF(n,j) = F(n+1,j).$$

Thus for $s \in \mathbb{Z}$,

$$N^s F(n,j) = F(n+s,j).$$

An *annihilating operator*, as the name implies, is an n-shift operator whose application to a polynomial results in zero. To obtain Laurent polynomials in N which represent n-shift annihilating operators for P_n, use `annihiloper`.

```
> annihiloper(q^(j^2)/spqfac(q,q,j),0,forward);
```

$$N^2 - N - q^{(n+1)}$$

returns a polynomial representing an annihilating operator for P_n with forward shifts, and

```
> annihiloper(q^(j^2)/spqfac(q,q,j),0,back);
```

$$1 - \frac{1}{N} - \frac{q^{(n-1)}}{N^2}$$

returns a Laurent polynomial representing an annihilating operator with backward shifts.

Since it is often desirable to execute all of the above procedures in sequence, `basic_calcs` allows the user to do this:

```
> basic_calcs(q^(j^2)/spqfac(q,q,j), 0, 18);
```

$$\phi_{18}(q) = \sum_{j=0}^{\infty} \frac{q^{(j^2)}}{(q,q)_j}$$

$$f_{18}(t) = \sum_{j=0}^{\infty} \frac{t^{(2j)} q^{(j^2)}}{(t,q)_{j+1}}$$

$$f_{18}(t) = \frac{1}{1-t} + \left(\frac{t^2 q}{1-t}\right) f_{18}(tq)$$

$$(1-t) f_{18}(t) = (1) + (t^2 q) f_{18}(tq)$$

$$P_n = P_{n-1} + P_{n-2} q^{(n-1)}$$

$$P_0 = 1$$

$$P_1 = 1$$

$$1 - \frac{1}{N} - \frac{q^{(n-1)}}{N^2}$$

$$N^2 - N - q^{(n+1)}$$

4.4 Conjecturing a Finitization

4.4.1 Calculating a Fermionic Form

As mentioned previously, the fermionic representation of P_n can be calculated by expanding the q-factorials in $f(q,t)$ using (2.10) and (2.11). The `fermipoly` procedure does this automatically:

```
> fermipoly( q^(j^2)/spqfac(q,q,j),0);
```

$$f(t) = \sum_{j,h=0}^{\infty} t^{(2j+h)} q^{(j^2)} \mathrm{gp}(j+h, j, 1)$$

Var to be eliminated, h

$$P_n = \sum_{j=0}^{\infty} q^{(j^2)} \mathrm{gp}(-j+n, j, 1)$$

The number of summations in the fermionic form of $P_n(q)$ corresponds to the number of q-factorials present in a given representation of $f(q,t)$. For example, consider

$$f(q,t) = \sum_{j=0}^{\infty} \frac{t^j q^{j^2}}{(t;q)_{2j+1}} = \sum_{j=0}^{\infty} \frac{t^j q^{j^2}}{(tq;q^2)_j (t;q^2)_{j+1}}. \tag{4.1}$$

If we use the representation in the center of (4.1), $P_n(q)$ will be represented as a double sum:

```
> fermipoly( q^(j^2)/ qfac(q,q^2,j) / spqfac(q^2,q^2,j));
```

$$f(t) = \sum_{j,h,k_1=0}^{\infty} t^{(j+h+k_1)} q^{(j^2+k_1)} gp(j-1-k_1, k_1, 2) gp(j+h, j, 2)$$

Variable to be eliminated, h

$$P_n = \sum_{j,k_1=0}^{\infty} q^{(j^2+k_1)} gp(j-1+k_1, k_1, 2) gp(n-k_1, j, 2).$$

We would probably prefer a single sum representation of $P_n(q)$, obtainable via the representation of $f(q,t)$ given on the righthand side of (4.1), but `fermipoly` does not allow the direct input of a q-factorial of the form $(q;q)_{2j+1}$. The
`fermipolylong` procedure allows for this. Note that `fermipolylong` requires the summand of $f(q,t)$ in its input:

```
> fermipolylong( t^j * q^(j^2) / spqfac(t,q,2*j+1));
```

$$f(t) = \sum_{j,h=0}^{\infty} t^{(j+h)} q^{(j^2)} gp(2j+h, 2j, 1)$$

Variable to be eliminated, h

$$P_n = \sum_{j=0}^{\infty} q^{(j^2)} gp(j+n, 2j, 1).$$

4.4.2 Conjecturing a Bosonic Form

We now need a Maple list containing $P_0(q), P_1(q), \ldots, P_N(q)$:

```
> Q:= polylist( q^(j^2)/spqfac(q,q,j),0, 30):
```

The final argument of 30 indicates that we want the first 30 elements of $P_0(q), P_1(q), \ldots$, i.e. the variable Q now contains $P_0(q), P_1(q), \ldots P_{29}(q)$. The user will probably want to suffix the procedure with a colon instead of a semi-colon to suppress the huge amount of output that would otherwise be generated by this procedure. However, it is necessary to look over at least part of this list, and for that purpose the `printpolyseq` procedure is included:

```
> printpolyseq(q^(j^2)/spqfac(q,q,j), 0, 8);
```

$$P_0 = 1$$
$$P_1 = 1$$
$$P_2 = q + 1$$
$$P_3 = q^2 + q + 1$$
$$P_4 = q^4 + q^3 + q^2 + q + 1$$
$$P_5 = q^6 + q^5 + 2q^4 + q^3 + q^2 + q + 1$$
$$P_6 = q^9 + q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1$$
$$P_7 = q^{12} + q^{11} + 2q^{10} + 2q^9 + 2q^8 + 2q^7 + 3q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q + 1$$

Since the product side of (1.1) is an instance of Jacobi's Triple Product Identity divided by $(q,q)_\infty$ and observing (2.8), it seems possible that a Gaussian polynomial will play a part in the bosonic representation of $P_n(q)$, but which one? It appears that $\deg(P_{2n}) = n^2$ and $\deg(P_{2n+1}) = n(n+1)$, so it seems reasonable to guess that the relevant Gaussian polynomials are $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ and $\begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}_q$ for the even and odd P_n's respectively. To check this guess, use the `search` procedure, which has the syntax

search (list, coefftype, index1, index2, p, terms, parity),
where *list* is a Maple list of polynomials; *coefftype* is "gp", "T0", "T1", "tau0", "U", "V", "t0", or "t1"; *index1* and *index2* are respectively the first and second indices of the q-bi/trinomial coëfficient indicated; p is the exponent of q in the base; *terms* is the number of terms in the list Q to to be checked; and *parity* is "even", "odd", or "all".

```
> Ev:= search(Q, gp, 2*n, n, 1, 10, even);
```

$$P_{2n} =$$
$$gp(2n,n,1) - q^2 gp(2n,n+2,1) - q^3 gp(2n,n+3,1) + (q^{11}+q^9) gp(2n,n+5,1)$$
$$-q^{21} gp(2n,n+7,1) - q^{24} gp(2n,n+8,1), \mathit{for}\ n < 10$$
$$Ev := [1, 0, -q^2, -q^3, 0, q^{11}+q^9, 0, -q^{21}, -q^{24}]$$

thus we have

$$P_n(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n+2 \end{bmatrix}_q - q^3 \begin{bmatrix} 2n \\ n+3 \end{bmatrix}_q + q^9 \begin{bmatrix} 2n \\ n+5 \end{bmatrix}_q + q^{11} \begin{bmatrix} 2n \\ n+5 \end{bmatrix}_q - \dots$$

which, using (2.4), can be rewritten as

$$P_n(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q - q^3 \begin{bmatrix} 2n \\ n+3 \end{bmatrix}_q + q^9 \begin{bmatrix} 2n \\ n-5 \end{bmatrix}_q + q^{11} \begin{bmatrix} 2n \\ n+5 \end{bmatrix}_q - \dots$$

Figuring the pattern that the exponents satisfy is aided by

```
> conjexps(Ev);
```

$$\frac{5}{2} j^2 + \frac{1}{2} j$$

```
> conjexpeven(Ev);
```

$$10j^2 + j$$

```
> conjexpodd(Ev);
```

$$10j^2 + 11j + 3$$

and so we conjecture that

$$P_{2m}(q) = \sum_{j=-\infty}^{\infty} q^{10j^2+j} \begin{bmatrix} 2m \\ m+5j \end{bmatrix}_q - q^{10j^2+11j+3} \begin{bmatrix} 2m \\ m+5j+2 \end{bmatrix}_q,$$

or equivalently,

$$P_{2m}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} 2m \\ m + \lfloor \frac{5j+1}{2} \rfloor \end{bmatrix}_q.$$

Using `search` with the option "odd" instead of "even" allows us to conjecture the corresponding representation for $P_{2m+1}(q)$. The results for even and odd n can be combined into

$$P_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n+5j+1}{2} \rfloor \end{bmatrix}_q,$$

the well-known representation for $P_n(q)$ due to Schur [1917]. Thus,

$$\sum_{j=0} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n+5j+1}{2} \rfloor \end{bmatrix}_q. \tag{4.2}$$

I have included several procedures to expedite the exploration of duality relationship explained in section 3.3. The `duallist` procedure accepts a list of polynomials as input and outputs a list containing the duals. For instance,

```
> duallist( [ 2q + 1, 3q^2 + 2q + 1, 4q^3 + 3q^2 + 2q + 1])
```

returns

$$[q+2, q^2+2q+3, q^4+2q^3+3q^2+4].$$

The `bosedual` procedure accepts as input a bosonic form involving a Gaussian polynomial, or a T_0 or T_1 trinomial coëfficient and finds the corresponding representation of the dual in terms of a Gaussian polynomial, or a t_0 or t_1 trinomial coëfficient. The syntax is as follows:

bosedual(summand, degree of P_n)

For example,

```
> bosedual((-1)^j * q^(j*(2*j+1))* gp(2*n+1, n+2*j+1, 1), n*(n+1))
```

returns

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2+j} gp(2n+1, n+2j+1, 1)$$

indicating that this polynomial is self-dual.

The `fermipolydual` procedure works just like the `fermipoly` procedure, except that it calculates the fermionic form for the dual of the polynomials indicated by the input, rather than for the polynomials themselves. Also, the final argument of the input is the degree of P_n.

For example, let us find the fermionic representation of the dual of identity (3.10):

```
> fermipolydual( qfac(-1,q^2,j) * q^(j^2)/ (qfac(q,q^2,j)
* spqfac(q^2,q^2,j)), 0, n^2);
```

$$\sum_{j,h,i_1,k_1=0}^{\infty} q^{(i_1+k_1+2i_1^2+2i_1h+2i_1k_1+h^2+2hk_1+k_1^2)} gp(n-i_1-h-k_1, i_1, 2)$$
$$gp(n-i_1-h-1, k_1, 2) gp(n-i_1-k_1, h, 2)$$

4.6 *n-Shift Operator Algebraic Procedures*

The `multops` and `rightdiv` procedures multiply and divide n-shift operators respectively. For example,

```
> A:= multops((1+q^2)*N - q^3, 1-N);
```

$$A := -q^3 + (-q^2 - 1)N^2 + (q^3 + q^2 + 1)N$$

```
> rightdiv(A,1-N);
```

$$(q^2 + 1)N - q^3$$

5 A User's Guide to the `recpf` Maple Package

5.1 *Setup and Initialization*

The `recpf` and `RRtools` packages are available for download from my web site, `http://www.math.rutgers.edu/~asills`. Copy these two files into the directory in which you intend to initiate your Maple session. Begin your Maple session as usual. Type

```
> read(recpf);
```

Maple responds with a welcome message, list of procedures, and automatically loads the `RRtools` package while loading the `recpf` package.

5.2 *Overview*

As mentioned earlier, all of the finite Rogers-Ramanujan type identities in §3 of Sills [2003] are theoretically provable by the q-Zeilberger algorithm [Zeilberger, 1991] or its multisum generalization [Wilf and Zeilberger, 1992]. Nonetheless, in practice, it was not possible to prove many of these identities using the RISC `qZeil` [Paule and Riese, 1997] or `qMultiSum` [Riese, 2003] Mathematica packages, due to complexity issues.

Thus, I offer an alternative. The method of recurrence proof is to show that a conjectured polynomial form satisfies a particular recurrence (and initial conditions). Accordingly, one applies a proposed annihilating operator to the conjectured bosonic form, and successively applies identities such as those found in §2.1, §2.2.2, and §2.2.3, until the expression simplifies to zero, thus demonstrating that the proposed annihilating operator is in fact an annihilating operator.

Note that the `recpf` package is *not* an automated proof package. The user must guide the proof at each step of the way. In this way, it is similar in spirit to Krattenthaler's `HYP` and `HYPQ` Mathematica packages [Krattenthaler, 1995], which follow the motto: "Do it by yourself!"

The advantage that `recpf` offers is that many of the tedious bookkeeping tasks associated with carrying out a recurrence proof on pencil and paper are made faster and more accurate. In this way, `recpf` is a fast and accurate "electronic secretary."

An explanation of each procedure is given below. Note that the finitization parameter *must* be n and the summation variable *must* be j, or the results computed may be incorrect.

`altsign` is a global boolean variable. If each term in an expression is to include the factor $(-1)^j$, this is encoded by setting `altsign` to *true*. The default value of `altsign` is *false*.

`applyshiftop (oper, summand)` applies the shift operator `oper` to the summand `summand`.

`asoe (oper, evensummand, oddsummand)`. When P_n has a different formula depending on the parity of n, one needs to verify the proposed annihilating operator for even and odd n separately. This procedure applies the shift operator `oper` to the summand whose formula for even n is given by `evensummand`, and for odd n by `oddsummand` in the case where n is assumed to be even.

`asoo (oper, evensummand, oddsummand)`. The analogous operation to the above for odd n.

`e0 (term, expr)` applies the relation

$$\mathrm{T}_1(L, q; q) = q^{L-A}\mathrm{T}_0(L, A; q) + \mathrm{T}_1(L, A+1; q) - q^{L+A+1}\mathrm{T}_0(L, A+1; q), \quad (5.1)$$

which is equivalent to (2.31) to the `term`-th term in the expression `expr` and leaves the other terms unchanged.

`eG0 (M, expr)` applies (2.5) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eG1 (M, expr)` applies (2.6) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eT0 (M, expr)` applies (2.26) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eT1 (M, expr)` applies (2.25) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eTrb0 (M, expr)` applies (2.28) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eTrb1 (M, expr)` applies (2.27) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eTrb28 (M, expr)` applies (2.29) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eTrb29 (M, expr)` applies (2.30) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eU0 (M, expr)` applies (2.35) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eU1 (M, expr)` applies (2.34) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`eV (M, expr)` applies (2.36) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`expandall (expr)` puts the exponents of each power of q appearing in each term of the expression `expr` into expanded form.

`gpsym (M, expr)` applies (2.4) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`neg(M, expr)` replaces j by $-j$ in the `M`-th term of the expression `expr`, which is legal since all bosonic forms are bilateral sums over j.

`printqmatch (expr)` prints an instance of a set of terms of `expr` with identical powers of q.

`printqmatchall (expr)` is like `printqmatch`, but prints all such instances instead of one.

`printT1 (expr)` prints a list of all terms in `expr` which contain the T_1 q-trinomial coëfficient.

`printU (expr)` prints a list of all terms in `expr` which contain the U function.

`shiftdown(M, expr)` replaces j by $j-1$ in the `M`-th term of the expression `expr`, which is legal since all Bosonic forms are bilateral sums over j.

`shiftup(M, expr)` replaces j by $j+1$ in the `M`-th term of the expression `expr`, which is legal since all Bosonic forms are bilateral sums over j.

`sym(M, expr)` replaces A by $-A$ in the q-trinomial $\mathrm{T}_0(L,A;q)$ or $\mathrm{T}_1(L,A;q)$ portion of the `M`-th term, and thus can be used to apply (2.38) or (2.39).

`splitU (M, expr)` applies (2.22) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

`splitV (M, expr)` applies (2.23) to the `M`-th term in the expression `expr` and leaves the other terms unchanged.

5.3 Demonstrations

5.3.1 Proof of a q-trinomial Identity

What follows is a transcription of a Maple session where identity 3.79-t of Sills [2003],

$$\sum_{j=0} q^{j^2} \begin{bmatrix} n+j \\ 2j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+2j} \mathrm{U}(n,5j;q), \tag{5.2}$$

is proved. It is assumed that the package `recpf` has already been loaded.

```
> op79:= annihiloper( q^(j^2)/ (qfac(q,q^2,j)
  * spqfac(q^2,q^2,j)), 0, back);
```

$$op79 := 1 - \frac{-q+1+q^{(2n-1)}}{N} - \frac{q}{N^2}$$

Note that the first argument of `annihiloper` is the limit as n tends to ∞ of the summand on the lefthand side of (5.2), i.e. the j-th term of the series which was finitized to obtain (5.2).

```
> altsign:= true:
> a1:= applyshiftop(op79, q^(10*j^2 +  2*j) * U(n, 5*j, 1));
```

$$\begin{aligned} a1 := &-q^{(1+10j^2+2j)}U(n-2,5j,1) - q^{(1+10j^2+2j)}U(n-1,5j,1) \\ &+q^{(10j^2+2j)}U(n-1,5j,1) - q^{(10j^2+2j-1+2n)}U(n-1,5j,1) \\ &-q^{(10j^2+2j)}U(n,5j,1) \end{aligned}$$

```
> a2 := eU1(2,a1);
```

$$a2 := -q^{(-2+10j^2+2j+2n)}U(n-2,5j,1) - q^{(10j^2-3j+n)}T1(n-2,5j-1,1)$$
$$-q^{(1+10j^2+7j+n)}T1(n-2,5j+2,1) + q^{(10j^2+2j)}U(n-1,5j,1)$$
$$-q^{(10j^2+2j-1+2n)}U(n-1,5j,1) - q^{(10j^2+2j)}U(n,5j,1)$$

```
> a3 := eU1(6,a2);
```

$$a3 := -q^{(-2+10j^2+2j+2n)}U(n-2,5j,1) - q^{(10j^2-3j+n)}T1(n-2,5j-1,1)$$
$$-q^{(1+10j^2+7j+n)}T1(n-2,5j+2,1) + q^{(10j^2-3j+n)}T1(n-1,5j-1,1)$$
$$-q^{(1+10j^2+7j+n)}T1(n-1,5j,1)$$

```
> a4 := eT1(4,a3);
```

$$a4 := -q^{(-2+10j^2+2j+2n)}U(n-2,5j,1) - q^{(1+10j^2+7j+n)}T1(n-2,5j+2,1)$$
$$+q^{(-2+10j^2+2j+2n)}T0(n-2,5j,1) + q^{(10j^2-8j+2n)}T0(n-2,5j-2,1)$$
$$-q^{(1+10j^2+7j+n)}T1(n-1,5j,1)$$

```
> a5 := eT1(5,a4);
```

$$a5 := -q^{(-2+10j^2+2j+2n)}U(n-2,5j,1) + q^{(-2+10j^2+2j+2n)}T0(n-2,5j,1)$$
$$+q^{(10j^2-8j+2n)}T0(n-2,5j-2,1) + q^{(2+10j^2+12j+2n)}T0(n-2,5j+3,1)$$
$$-q^{(-2+10j^2+2j+2n)}T0(n-2,5j+1,1)$$

```
> a6 := splitU(1,a5);
```

$$a6 := q^{(10j^2-8j+2n)}T0(n-2,5j-2,1) + q^{(2+10j^2+12j+2n)}T0(n-2,5j+3,1)$$

```
> a7 := shiftdown(2,a6);
```

$$a7 := 0$$

5.3.2 Proof of a Finite Second Rogers-Ramanujan Identity

What follows is a transcription of a Maple session where the following finitization of (1.2) is proved.

$$\sum_{j=0} q^{j(j+1)} \begin{bmatrix} n-j \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \begin{bmatrix} n+1 \\ \lfloor \frac{n+5j+3}{2} \rfloor \end{bmatrix}_q \tag{5.3}$$

It is assumed that the package `recpf` has already been loaded.

```
> fermipoly( q^(j^2+j)/spqfac(q,q,j),0);
```

$$f(t) = \sum_{j,h=0}^{\infty} t^{(2j+h)} q^{(j^2+j)} gp(j+h, j, 1)$$

$$\textit{Variable to be eliminated, } h$$

$$P_n = \sum_{j=0}^{\infty} q^{(j^2+j)} gp(-j+n, j, 1)$$

```
> altsign:=false:
> op14:= annihiloper( q^(j^2+j)/spqfac(q,q,j), 0, forward);
```

$$op14 := N^2 - N - q^{(n+2)}$$

```
> a1 := asoe(op14, q^(10*j^2 +  3*j    ) * gp(2*n+1, n+5*j+1, 1)
               -q^(10*j^2 + 13*j + 4) * gp(2*n+1, n+5*j+4, 1),
                q^(10*j^2 +  3*j    ) * gp(2*n+2, n+5*j+2, 1)
               -q^(10*j^2 + 13*j + 4) * gp(2*n+1, n+5*j+4, 1));
```

$$\begin{aligned} a1 := &-gp(2m+1, m+5j+1, 1) q^{(2m+2+10j^2+3j)} \\ &+gp(2m+1, m+5j+4, 1) q^{(2m+6+10j^2+13j)} \\ &+q^{(10j^2+3j)} gp(2m+3, m+5j+2, 1) \\ &-q^{(10j^2+13j+4)} gp(2m+3, m+5+5j, 1) \\ &-q^{(10j^2+3j)} gp(2m+2, m+5j+2, 1) \\ &+q^{(10j^2+13j+4)} gp(2m+2, m+5j+4, 1) \end{aligned}$$

```
> a2:= eG0(3,a1)
```

$$\begin{aligned} a2 := &-gp(2m+1, m+5j+1, 1) q^{(2m+2+10j^2+3j)} \\ &+gp(2m+1, m+5j+4, 1) q^{(2m+6+10j^2+13j)} \\ &+q^{(10j^2-2j+m+1)} gp(2m+2, m+5j+1, 1) \\ &-q^{(10j^2+13j+4)} gp(2m+3, m+5+5j, 1) \\ &+q^{(10j^2+13j+4)} gp(2m+2, m+5j+4, 1) \end{aligned}$$

```
> a3:= eG1(4,a2)
```

$$\begin{aligned} a3 := &-gp(2m+1, m+5j+1, 1) q^{(2m+2+10j^2+3j)} \\ &+gp(2m+1, m+5j+4, 1) q^{(2m+6+10j^2+13j)} \\ &+q^{(10j^2-2j+m+1)} gp(2m+2, m+5j+1, 1) \\ &-q^{(10j^2+18j+9+m)} gp(2m+2, m+5+5j, 1) \end{aligned}$$

```
> a4:= shiftdown(4,a3)
```

$$
\begin{aligned}
a4 := &-gp(2m+1, m+5j+1, 1)q^{(2m+2+10j^2+3j)}\\
&+gp(2m+1, m+5j+4, 1)q^{(2m+6+10j^2+13j)}\\
&+q^{(10j^2-2j+m+1)}gp(2m+2, m+5j+1, 1)\\
&-q^{(10j^2-2j+m+1)}gp(2m+2, m+5j, 1)
\end{aligned}
$$

```
> a5:= eG1(3,a4)
```

$$
\begin{aligned}
a5 := &\, gp(2m+1, m+5j+4, 1)q^{(2m+6+10j^2+13j)}\\
&+q^{(10j^2-2j+m+1)}gp(2m+1, m+5j+1, 1)\\
&-q^{(10j^2-2j+m+1)}gp(2m+2, m+5j, 1)
\end{aligned}
$$

```
> a6:= eG0(3,a5)
```

$$
\begin{aligned}
a5 := &\, gp(2m+1, m+5j+4, 1)q^{(2m+6+10j^2+13j)}\\
&+q^{(10j^2-7j+2m+3)}gp(2m+1, m+5j-1, 1)
\end{aligned}
$$

```
> a7:= shiftdown(1,a6)
```

$$a7 := 0$$

Thus, the operator `op14` indeed annihilates the conjectured bosonic form for even n. The proof for odd n is similar; one must of course use `asoo` instead of `asoe` in the beginning of the proof.

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