Lecture hall sequences, q-series, and asymmetric partition identities

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Abstract

We use generalized lecture hall partitions to discover a new pair of q-series identities. These identities are unusual in that they involve partitions into parts from asymmetric residue classes, much like the little Göllnitz partition theorems. We derive a two-parameter generalization of our identities that, surprisingly, gives new analytic counterparts of the little Göllnitz theorems. Finally, we show that the little Göllnitz theorems also involve "lecture hall sequences", that is, sequences constrained by the ratio of consecutive parts.

Keywords: lecture hall partitions, q-series identities, q-Gauss summation, Göllnitz partition theorems

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1 Introduction

In this paper we illustrate the role that can be played by sequences constrained by the *ratio* of consecutive parts in interpreting and discovering q-series identities.

Let $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ and $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$. We derive the identities

$$\sum_{j=0}^{\infty} q^{j(3j-1)/2}\frac{(q^2;q^6)_j}{(q;q)_{3j}} = \frac{1}{(q;q^3)_\infty(q^5;q^6)_\infty} \tag{1}$$

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and

$$\sum_{j=0}^{\infty} q^{j(3j+1)/2} \frac{(q^4;q^6)_j}{(q;q)_{3j+1}} \quad = \quad \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty} \tag{2}$$

by showing that both sides of (1) count (by weight) the finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots \tag{3}$$

and both sides of (2) count the finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \cdots . \tag{4}$$

Contrast these with Euler's odd-distinct partition identity

$$\sum_{j=0}^{\infty} q^{j(j+1)/2} \frac{1}{(q;q)_j} \quad = \quad \frac{1}{(q;q^2)_\infty}, \tag{5}$$

both sides of which count the finite sequences of positive integers satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{1} > \frac{\lambda_3}{1} > \frac{\lambda_4}{1} > \cdots . \tag{6}$$

Our methods combine results on *lecture hall partitions* from [3], on *sequences constrained by the ratio of successive parts* from [5], and *combinatorial reciprocity* [13]. In Section 2, we use "lecture hall" methods to show that the right-hand sides of (1) and (2) count solutions to (3) and (4), respectively. In Section 3, we show that the left-hand sides of (1) and (2) also count solutions to (3) and (4), using results from [5].

In Section 4, we refine the counting arguments in Sections 2 and 3 to derive a two-parameter q-series identity, $I(a,q)$, generalizing (1) and (2). We show in Section 5 that $I(a,q)$ can be obtained as a specialization of the q-Gauss summation [6].

Say that a set, $R = \{r_1, r_2, \ldots, r_k\}$, of residue classes modulo m, is *symmetric* if

$$R = \{m - r_1, m - r_2, \ldots, m - r_k\}.$$

It is noteworthy that the infinite products appearing in (1) and (2) are generating functions for partitions into parts from residue classes modulo 6 which are *not* symmetric.

Most well-known partition theorems involve symmetric residue classes, e.g. the Rogers-Ramanujan identities and the Gordon and Bressoud generalizations thereof [8, 4], Schur's 1926 partition theorem related to the modulus 6 [11], and the Göllnitz-Gordon identities (Göllnitz [7, pp. 162–163, Satz 2.1 and 2.2], Gordon [9, p. 741, Thms. 2 and 3]). From a q-series perspective, this is a consequence of the fact that the relevant generating functions are modular forms (up to multiplication by a trivial factor).

Perhaps the best known partition identities involving *asymmetric* residue classes are a pair of identities known as "Göllnitz's little partition theorems" [7, pp. 166–167, Satz 2.3 and 2.4] and the "big" Göllnitz partition theorem related to the modulus 12 ([7, p. 175, Satz 4.1]; cf. [1, p. 37, Thm. 1]). In Section 6, we show that an appropriate specialization of $I(a,q)$ gives a different view of the infinite products appearing in (the analytic forms of) Göllnitz's little partition theorems. Furthermore, we show that the little Göllnitz theorems can be alternately viewed as statements about partitions constrained by the *ratio* of consecutive parts.

2 The "lecture hall" approach

The purpose of this section is to show that the right-hand sides of identities (1) and (2) count solutions to the inequalities (3) and (4), respectively. We begin with a theorem of Bousquet-Mélou and Eriksson [3] about (k, l) *sequences.*

Given two integers k and l greater than one, the (k, l)*-sequence* $\{a_n^{(k,l)}\}$ is defined in [3] for $n \geq 0$ by the following recurrence:

$$\begin{aligned} a_{2i}^{(k,l)} &= l a_{2i-1}^{(k,l)} - a_{2i-2}^{(k,l)}, \\ a_{2i+1}^{(k,l)} &= k a_{2i}^{(k,l)} - a_{2i-1}^{(k,l)}, \end{aligned}$$

for $i \geq 1$, with initial conditions $a_0^{(k,l)} = 0$ and $a_1^{(k,l)} = 1$.

Let $L_n^{(k,l)}$ be the set of nonnegative integer sequences λ of length n satisfying,

$$\frac{\lambda_1}{a_n^{(k,l)}} \geq \frac{\lambda_2}{a_{n-1}^{(k,l)}} \geq \ldots \geq \frac{\lambda_n}{a_1^{(k,l)}} \geq 0.$$

The following was shown in [3].

Theorem 1 (The (k, l)-Lecture Hall Theorem) *The generating function for $L_n^{(k,l)}$ is given by*

$$G_n^{(k,l)}(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a_i^{(k,l)} + a_{i-1}^{(l,k)}}} \qquad \text{if } n \text{ is even}$$

$$G_n^{(k,l)}(q) = \prod_{i=1}^{n} \frac{1}{1 - q^{a_i^{(l,k)} + a_{i-1}^{(k,l)}}} \qquad \text{if } n \text{ is odd}$$

When $k \geq 2$ and $l \geq 2$, the sequence $\{a_n^{(k,l)}\}$ is strictly increasing. When $k = 1$ or $l = 1$, the sequence $\{a_n^{(k,l)}\}$ is not monotone and, when $kl < 4$ some terms will be negative. Nevertheless, we make the following observation, and prove it in Appendix 1.

Observation 1 *The (k, l)-Lecture Hall Theorem remains true when $k = 1$ or $l = 1$, as long as $kl \geq 4$.*

For our application, consider the sequences:

$$\begin{aligned} a^{(1,4)} &= 0,\ 1,\ 4,\ 3,\ 8,\ 5,\ 12,\ 7, \ldots; \\ a^{(4,1)} &= 0,\ 1,\ 1,\ 3,\ 2,\ 5,\ 3,\ 7, \ldots. \end{aligned}$$

Then

$$\begin{aligned} a_{2i+1}^{(1,4)} &= 2i + 1; \qquad & a_{2i}^{(1,4)} &= 4i; \\ a_{2i+1}^{(4,1)} &= 2i + 1; \qquad & a_{2i}^{(4,1)} &= i. \end{aligned}$$

So, by definition of $G_n^{(k,l)}(q)$,

$$G_{2k}^{(1,4)}(q) = \prod_{i=0}^{k-1} \frac{1}{(1 - q^{a_{2i+1}^{(1,4)} + a_{2i}^{(4,1)}})(1 - q^{a_{2i+2}^{(1,4)} + a_{2i+1}^{(4,1)}})} = \prod_{i=0}^{k-1} \frac{1}{(1 - q^{3i+1})(1 - q^{6i+5})}$$

and

$$\lim_{k\to\infty} G_{2k}^{(1,4)}(q) \quad = \quad \frac{1}{(q;q^3)_\infty (q^5;q^6)_\infty}, \tag{7}$$

giving the right-hand side of (1). On the other hand, by Theorem 1, $G_{2k}^{(1,4)}(q)$ is the generating function for $L_{2k}^{(1,4)}$, the set of sequences satisfying

$$L_{2k}^{(1,4)}: \quad \frac{\lambda_1}{4k} \geq \frac{\lambda_2}{2k-1} \geq \frac{\lambda_3}{4(k-1)} \geq \frac{\lambda_4}{2k-3} \geq \ldots \geq \frac{\lambda_{2k-1}}{4} \geq \frac{\lambda_{2k}}{1} \geq 0.$$

Note that

$$\lim_{k\to\infty} \frac{a_{2k}^{(1,4)}}{a_{2k-1}^{(1,4)}} = \frac{4k}{2k-1} = 2$$

and

$$\lim_{k\to\infty} \frac{a_{2k+1}^{(1,4)}}{a_{2k}^{(1,4)}} = \frac{2k+1}{4k} = \frac{1}{2}$$

so $\lim_{k\to\infty} L_{2k}^{(1,4)}$ is the set of sequences satisfying the constraints (3), whose generating function must therefore be (7).

Similarly, by definition of $G_n^{(k,l)}(q)$,

$$\begin{aligned} G_{2k+1}^{(1,4)}(q) \quad &= \quad \frac{1}{1-q} \prod_{i=1}^{k} \frac{1}{(1-q^{a_{2i}^{(4,1)}+a_{2i-1}^{(1,4)}})(1-q^{a_{2i+1}^{(4,1)}+a_{2i}^{(1,4)}})} \\ &= \quad \frac{1}{1-q} \prod_{i=1}^{k} \frac{1}{(1-q^{3i-1})(1-q^{6i+1})} \end{aligned}$$

and

$$\lim_{k\to\infty} G_{2k+1}^{(1,4)}(q) \quad = \quad \frac{1}{(q^2;q^3)_\infty (q;q^6)_\infty}, \tag{8}$$

giving the right-hand side of (2).

On the other hand, by Theorem 1, $G_{2k+1}^{(1,4)}(q)$ is the generating function for $L_{2k+1}^{(1,4)}$, the sequences satisfying

$$L_{2k+1}^{(1,4)}: \quad \frac{\lambda_1}{2k+1} \geq \frac{\lambda_2}{4k} \geq \frac{\lambda_3}{2k-1} \geq \frac{\lambda_4}{4(k-1)} \geq \ldots \geq \frac{\lambda_{2k}}{4} \geq \frac{\lambda_{2k+1}}{1} \geq 0.$$

As $k \to \infty$, $L_{2k+1}^{(1,4)}$ becomes the set of sequences satisfying the constraints (4) and thus their generating function is given by (8).

3 The "enumerative combinatorics" approach

In this section, we use results from [5] to show that the left-hand sides of identities (1) and (2) count the integer solutions to the inequalities (3) and (4), respectively. Define $[n]_q$ by

$$[n]_q := (1-q^n)/(1-q).$$

The following is shown in [5] (we include a self-contained proof in Appendix 2).

Theorem 2 *Let $s_1, s_2, \ldots, s_k$ be a sequence of positive integers satisfying the condition $s_i = 1$ or $s_{i+1} = 1$ for $1 \le i \le k-1$. Then the generating for the nonnegative integer sequences λ satisfying*

$$\frac{\lambda_1}{s_1} \ge \frac{\lambda_2}{s_2} \ge \cdots \ge \frac{\lambda_k}{s_k} \ge 0$$

is

$$F(q) = \sum_{\lambda} q^{|\lambda|} = \frac{\prod_{i=2}^{k-1}(1 + q^{b_i}([s_{i+1}]_q - 1))}{\prod_{i=1}^{k}(1 - q^{b_i})},$$

where $b_1 = 1$ and $b_i = s_1 + \ldots + s_i$ for $i > 1$.

Note that if $s_1 = s_2 = \cdots = s_k = 1$, then $F(q)$ is the generating function for ordinary partitions with at most k parts.

We first apply Theorem 2 to the sequence $s = (2, 1, 2, 1, \ldots)$. Note that $b_{2j} = 3j$, $b_{2j+1} = 3j + 2$, so

$$\begin{aligned} 1 + q^{b_{2j}}([s_{2j+1}]_q - 1) = 1 + q^{3j}([2]_q - 1)) &= 1 + q^{3j+1} \\ 1 + q^{b_{2j+1}}([s_{2j+2}]_q - 1) = 1 + q^{3j+2}([1]_q - 1) &= 1. \end{aligned}$$

Thus, for $s = (2, 1, 2, 1, \ldots)$, the generating function for the sequences satisfying

$$\frac{\lambda_1}{2} \ge \frac{\lambda_2}{1} \ge \frac{\lambda_3}{2} \ge \frac{\lambda_4}{1} \ge \cdots \ge \frac{\lambda_n}{s_n} \ge 0 \tag{9}$$

is

$$\begin{aligned} f_n(q) &= \frac{(-q^4; q^3)_{\lfloor (n-1)/2 \rfloor}}{(1-q)(q^3; q^3)_{\lfloor n/2 \rfloor}(q^5; q^3)_{\lfloor (n+1)/2 \rfloor}} \\ &= \frac{(1+q)(-q^4; q^3)_{\lfloor (n-1)/2 \rfloor}}{(q^3; q^3)_{\lfloor n/2 \rfloor}(q^2; q^3)_{\lfloor (n+1)/2 \rfloor}} \\ &= \frac{(-q; q^3)_{\lfloor (n+1)/2 \rfloor}}{(q^3; q^3)_{\lfloor n/2 \rfloor}(q^2; q^3)_{\lfloor (n+1)/2 \rfloor}} \\ &= \frac{(q^2; q^6)_{\lfloor (n+1)/2 \rfloor}}{(q; q)_{\lfloor (3n+1)/2 \rfloor}} \end{aligned}$$

The constraints (9) define a simplicial cone, so Stanley's reciprocity theorem [13] can be used to compute, from $f_n(q)$, the generating function for those integer points *interior* to the cone. Specifically, the generating function for those integer sequences λ satisfying the *strict* constraints

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots > \frac{\lambda_n}{s_n} > 0$$

is given by

$$h_n(q) = (-1)^n f_n(1/q) = \begin{cases} q^{(3n^2+10n)/8} f_n(q) & n \text{ even} \\ q^{(3n^2+4n+1)/8} f_n(q) & n \text{ odd.} \end{cases} \tag{10}$$

Finally, the generating function for the integer sequences λ satisfying (3) can now be obtained by summing $h_n(q)$ in (10) over all $n \geq 0$:

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(q) &= 1 + \sum_{k=1}^{\infty} (h_{2k-1} + h_{2k}) \\
&= 1 + \sum_{k=1}^{\infty} \left(\frac{q^{(3(2k-1)^2+4(2k-1)+1)/8}(q^2;q^6)_k}{(q;q)_{3k-1}} + \frac{q^{(3(2k)^2+10(2k))/8}(q^2;q^6)_k}{(q;q)_{3k}} \right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{q^{k(3k-1)/2}(q^2;q^6)_k}{(q;q)_{3k}},
\end{aligned}$$

which agrees with the left-hand side of equation (1).

We proceed similarly to find the generating function of the solutions of (4). In this case, we apply Theorem 2 to the sequence $s' = (1, 2, 1, 2, ...)$. For this sequence, $b_{2j} = 3j$, $b_{2j+1} = 3j + 1$, so

$$\begin{aligned}
1 + q^{b_{2j}}([s'_{2j+1}]_q - 1) = 1 + q^{3j}([1]_q - 1)) &= 1 \\
1 + q^{b_{2j+1}}([s'_{2j+2}]_q - 1) = 1 + q^{3j+1}([2]_q - 1) &= 1 + q^{3j+2}.
\end{aligned}$$

Thus, for $s' = (1, 2, 1, 2, ...)$, the generating function for the sequences satisfying

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \frac{\lambda_3}{1} \geq \frac{\lambda_4}{2} \geq \cdots \geq \frac{\lambda_n}{s'_n} \geq 0 \tag{11}$$

is

$$\begin{aligned}
f'_n(q) &= \frac{(-q^5;q^3)_{\lfloor (n-2)/2 \rfloor}}{(q;q^3)_{\lfloor (n+1)/2 \rfloor}(q^3;q^3)_{\lfloor (n-1)/2 \rfloor}} \\
&= \frac{(q^{10};q^6)_{\lfloor (n-2)/2 \rfloor}}{(q;q^3)_{\lfloor (n+1)/2 \rfloor}(q^3;q^3)_{\lfloor (n-1)/2 \rfloor}(q^5;q^3)_{\lfloor (n-2)/2 \rfloor}} \\
&= \frac{(q^2;q^6)_{\lfloor n/2 \rfloor}}{(q;q)_{\lfloor 3n/2 \rfloor}}
\end{aligned}$$

Again, by the reciprocity theorem [13], the generating function for those integer sequences λ satisying the *strict* constraints

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \cdots > \frac{\lambda_n}{s'_n} > 0$$

is given by

$$h'_n(q) = (-1)^n f'_n(1/q) = \begin{cases} q^{k(3k+1)/2} f'_{2k}(q) & n = 2k \\ q^{(k+2)(3k+1)/2} f'_{2k+1}(q) & n = 2k+1 \ . \end{cases} \tag{12}$$

Finally, the generating function for the integer sequences λ satisfying (4) can now be obtained by summing $h'_n(q)$ in (12) over all $n \geq 0$:

$$\begin{aligned}\sum_{n=0}^{\infty} h'_n(q) &= \sum_{k=0}^{\infty}(h'_{2k} + h'_{2k+1}) \\ &= \sum_{k=0}^{\infty}\left(\frac{q^{k(3k+1)/2}(q^4;q^6)_k}{(q;q)_{3k}} + \frac{q^{k(3k+1)/2}(q^2;q^6)_k}{(q;q)_{3k+1}}\right) \\ &= \sum_{k=0}^{\infty}\frac{q^{(k+1)(3k+2)/2}(q^4;q^6)_k}{(q;q)_{3k+1}}\end{aligned}$$

which agrees with the left-hand side of eq. (2).

4 A refinement

Define the even and odd weight of a sequence $\lambda = (\lambda_1, \lambda_2, ...)$ by

$$|\lambda|_o = \lambda_1 + \lambda_3 + \cdots; \qquad |\lambda|_e = \lambda_2 + \lambda_4 + \cdots;$$

and define

$$G_n^{(k,l)}(x,y) = \sum_{\lambda \in L_n^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e}.$$

As we indicate in Appendix 1, what Bousquet-Mélou and Eriksson proved in [3] was the following: The generating function for $L_n^{(k,l)}$ is given by

$$G_n^{(k,l)}(x,y) = \prod_{i=1}^{n} \frac{1}{1 - x^{a_i^{(k,l)}} y^{a_{i-1}^{(l,k)}}} \quad \text{if } n \text{ is even}$$

$$G_n^{(k,l)}(x,y) = \prod_{i=1}^{n} \frac{1}{1 - x^{a_i^{(l,k)}} y^{a_{i-1}^{(k,l)}}} \quad \text{if } n \text{ is odd}$$

So

$$\lim_{k\to\infty} G_{2k}^{(1,4)}(x,y) = \frac{1}{(x;x^2y)_\infty (x^4y;x^4y^2)_\infty} = \frac{(-x;x^2y)_\infty}{(x^2;x^2y)_\infty}. \tag{13}$$

Similarly, the counting method of Section 3 also admits an x,y-refinement. From the bijective proof Theorem 2 that appears in Appendix 2, it can be checked that the 2-variable version of the generating function for the sequences λ satisfying (9) is

$$f_n(x,y) = \sum_{\lambda} x^{|\lambda|_o} y^{|\lambda|_e} = \frac{(-x;x^2y)_{\lfloor (n+1)/2 \rfloor}}{(x^2y;x^2y)_{\lfloor n/2 \rfloor}(x^2;x^2y)_{\lfloor (n+1)/2 \rfloor}}.$$

Proceeding as in Section 3, using reciprocity,

$$h_n(x,y) = (-1)^n f_n(1/x, 1/y),$$

and summing over all n, gives another expression for the generating function of (3):

$$\sum_{n=0}^{\infty} h_n(x,y) = 1 + \sum_{k=1}^{\infty} (h_{2k-1}(x,y) + h_{2k}(x,y))$$
$$= \sum_{j=0}^{\infty} \frac{x^{j^2} y^{j(j-1)/2} (-x; x^2y)_j}{(x^2; x^2y)_j (x^2y; x^2y)_j} \tag{14}$$

Since both (13) and (14) count (3), we have the following.

Theorem 3

$$\sum_{j=0}^{\infty} \frac{x^{j^2} y^{j(j-1)/2} (-x; x^2y)_j}{(x^2; x^2y)_j (x^2y; x^2y)_j} = \sum_{\lambda} x^{\lambda_1+\lambda_3+\cdots} y^{\lambda_2+\lambda_4+\cdots} = \frac{(-x; x^2y)_\infty}{(x^2; x^2y)_\infty}$$

where the second sum is over all positive integer sequences λ satisyfing

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \cdots . \tag{15}$$

Setting $x = -a$ and $y = q/a^2$ gives the following identity, which we refer to as $I(a, q)$.

Corollary 1

$$I(a,q) := \sum_{n=0}^{\infty} \frac{(a;q)_n (-a)^n q^{\binom{n}{2}}}{(a^2;q)_n (q;q)_n} = \frac{(a;q)_\infty}{(a^2;q)_\infty}. \tag{16}$$

As an alternative to reciprocity, we could explain the sum side of (14) combinatorially.

5 Deriving the identities from the q-Gauss summation

Recall Heine's q-Gauss summation [6, Eq. (II.8)]:

$$H(a,b,c;q) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/(ab);q)_\infty}. \tag{17}$$

Note that as $b \to \infty$, we have $(b;q)_n/b^n \to (-1)^n q^{\binom{n}{2}}$ and $(x/b;q)_\infty \to 1$, so (17) becomes

$$H(a,\infty,c;q) = \sum_{n=0}^{\infty} \frac{(a;q)_n (-c/a)^n q^{\binom{n}{2}}}{(c;q)_n (q;q)_n} = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{18}$$

Thus

$$H(a,\infty,a^2;q) = \sum_{n=0}^{\infty} \frac{(a;q)_n (-a)^n q^{\binom{n}{2}}}{(a^2;q)_n (q;q)_n} = \frac{(a;q)_\infty}{(a^2;q)_\infty},$$

which is (16). So

$$I(a,q) = H(a,\infty,a^2;q).$$

Then,

$$I(-q,q^3) \quad = \quad \sum_{n=0}^{\infty} \frac{(-q;q^3)_n \; q^{n(3n-1)/2}}{(q^2;q^3)_n (q^3;q^3)_n} \quad = \quad \frac{(-q;q^3)_\infty}{(q^2;q^3)_\infty},$$

which is equivalent to (1).

On the other hand, identity (2) is equivalent to

$$\frac{I(-q^2,q^3)}{1-q} \quad = \quad \sum_{n=0}^{\infty} \frac{(-q^2;q^3)_n q^{n(3n+1)/2}}{(q;q^3)_{n+1} (q^3;q^3)_n} \quad = \quad \frac{(-q^2;q^3)_\infty}{(q;q^3)_\infty}.$$

It is interesting to note that (5) follows in a similar way from (18):

$$H(\infty,\infty,q;q^2) \quad = \quad \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(q;q)_{2n}} \quad = \quad \frac{1}{(q;q^2)_\infty}. \tag{19}$$

Equation (19) is in fact equivalent to an identity appearing in Slater's compendium of Rogers-Ramanujan type identities [12, p. 157, Eq. (52)].

Observe that

$$\frac{q^{n(2n-1)}}{(q;q)_{2n}} \quad = \quad \frac{q^{(2n-1)(2n)/2}}{(q;q)_{2n-1}} + \frac{q^{(2n)(2n+1)/2}}{(q;q)_{2n}},$$

so that each term of the sum in (19) is the sum of two successive terms of the sum in (5).

6 Lebesgue's identity and a new view of Göllnitz's "little" partition theorems

The infinite products appearing in (1) and (2) enumerate partitions whose parts belong to the residue classes $\{1,4,5\}$ modulo 6 and $\{1,2,5\}$ modulo 6, respectively. It is noteworthy that these residue classes are *not* symmetric modulo 6, since most well-known partition theorems involve symmetric residue classes.

Two of the best known partition identities involving *asymmetric* residue classes are known as "Göllnitz's little partition theorems" [7, pp. 166–167, Satz 2.3 and 2.4].

Theorem 4 (Göllnitz) *The number of partitions of N into parts differing by at least* 2 *and no consecutive odd parts equals the number of partitions of N into parts congruent to* 1, 5 *or* 6 *modulo* 8.

Theorem 5 (Göllnitz) *The number of partitions of N into parts differing by at least* 2*, no consecutive odd parts, and no ones equals the number of partitions of N into parts congruent to* 2, 3 *or* 7 *modulo* 8.

It is well known that the analytic counterparts to Theorems 4 and 5 are special cases of an identity due to V. A. Lebesgue ([10]; cf. [2, p. 21, Cor. 2.7]):

$$L(a,q) \quad := \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(a;q)_n}{(q;q)_n} \quad = \quad \frac{(aq;q^2)_\infty}{(q;q^2)_\infty}. \tag{20}$$

The analytic counterpart to Theorem 4 is [7, Eq. (2.22)]

$$L(-q^{-1}, q^2) \quad = \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1}; q^2)_n}{(q^2; q^2)_n} \quad = \quad \frac{1}{(q; q^4)_\infty (q^6; q^8)_\infty}, \tag{21}$$

while that of Theorem 5 is [7, Eq. (2.24)]

$$L(-q, q^2) \quad = \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q; q^2)_n}{(q^2; q^2)_n} \quad = \quad \frac{1}{(q^2; q^8)_\infty (q^3; q^4)_\infty}. \tag{22}$$

However, it may not have been observed previously that *the infinite products in* (21) *and* (22) *also arise as special cases of the* q*-Gauss sum* (17), via $I(a, q)$. By appropriate specialization, we obtain

$$I(-q, q^4) \quad = \quad \sum_{n=0}^{\infty} \frac{q^{2n^2-n}(-q; q^4)_n}{(q^2; q^2)_{2n}} \quad = \quad \frac{1}{(q; q^4)_\infty (q^6; q^8)_\infty} \tag{23}$$

and

$$\frac{I(-q^3, q^4)}{1-q^2} \quad = \quad \sum_{n=0}^{\infty} \frac{q^{2n^2+n}(-q^3; q^4)_n}{(q^2; q^2)_{2n+1}} \quad = \quad \frac{1}{(q^2; q^8)_\infty (q^3; q^4)_\infty}. \tag{24}$$

Finally, we observe that Göllnitz's little partition theorems can be alternately viewed as theorems about partitions constrained by the ratio of consecutive parts and give a combinatorial interpretation of (23).

Observation 2 *The set of partitions of* N *into parts differing by at least* 2 *and no consecutive odd parts is the same as the set of finite sequences of positive integers* $\lambda_1, \lambda_2, \ldots$ *of weight* N *satisfying*

$$\left\lfloor \frac{\lambda_i}{2} \right\rfloor \quad > \quad \left\lceil \frac{\lambda_{i+1}}{2} \right\rceil.$$

7 Suggestions for further study

Can we derive other series-product identities from the lecture hall approach, via Theorem 1 and Observation 1? Although these tools produce a "product side" for any positive integers (k, j) with $kl \geq 4$, deriving a "sum side" from the ratio characterization is more difficult. As shown in [3], the limiting form of Theorem 1 gives rise to the following ratios between consecutive parts: $(kl + \sqrt{kl(kl-1)})/(2k)$ and $(kl + \sqrt{kl(kl-1)})/(2l)$. These ratios are rational only if either $\{k, l\} = \{1, 4\}$, (the case considered in this paper) or $k = l = 2$ (giving ratio 1, the familiar case of distinct parts).

Are there other classical partition theorems that can be re-interpreted as statements about partitions constrained by the ratio of consecutive parts? For example, Gordon's combinatorial interpretation [9, p. 741, Thms. 2 and 3] of the the Göllnitz-Gordon identities involves partitions into parts differing by at least 2 and no consecutive even parts. Such partitions can be alternatively characterized as the set of finite sequences of positive integers $\lambda_1, \lambda_2, \ldots$ satisfying, for each i,

$$\left\lfloor \frac{\lambda_i + 1}{2} \right\rfloor \quad > \quad \left\lceil \frac{\lambda_{i+1} + 1}{2} \right\rceil.$$

We expect to find other examples. What can be learned from these re-interpretations?

8 Appendix 1: Proof of Observation 1

To verify that the (k,l)-Lecture Hall Theorem remains true for all *positive* k,l satisfying $kl \geq 4$, we first observe that these conditions are necessary and sufficient to guarantee that $a_n^{(k,l)}$ is positive for all $n \geq 1$. When $kl \geq 4$, each of the sequences $\{a_{2i}^{(k,l)}\}_{i\geq 0}$, $\{a_{2i+1}^{(k,l)}\}_{i\geq 0}$ satisfies the recurrence

$$w_i = (kl-2)w_{i-1} - w_{i-2},$$

and, with their respective initial conditions, are strictly increasing. On the other hand, it can be checked that negative elements appear in the sequence when $(k,l) \in \{(1,1),(1,2),(2,1),(1,3),(3,1)\}$.

We then outline the clever combinatorial/algebraic approach of Bousquet-Mélou and Eriksson in [3], to illustrate that in order for Theorem 1 to hold, it is not necessary that $a_1^{(k,l)}, \ldots, a_n^{(k,l)}$ be weakly increasing, rather only that all terms are positive.

Define the even and odd weight of a sequence $\lambda = (\lambda_1, \lambda_2, ...)$ by

$$|\lambda|_o = \lambda_1 + \lambda_3 + \cdots ; \qquad |\lambda|_e = \lambda_2 + \lambda_4 + \cdots ;$$

and define

$$G_n^{(k,l)}(x,y) = \sum_{\lambda \in L_n^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e}.$$

The strategy is to show that the following recurrence from [3] holds for all *positive* k,l satisfying $kl \geq 4$.

$$G_n^{(k,l)}(x,y) \quad = \quad \begin{cases} G_{n-1}^{(k,l)}(x^l y, x^{-1})/(1-x) & \text{if } n \text{ is even} \\ \\ G_{n-1}^{(k,l)}(x^k y, x^{-1})/(1-x) & \text{if } n \text{ is odd,} \end{cases} \tag{25}$$

with initial condition $G_0^{(k,l)}(x,y) = 1$. Using the recursive definition of $a_n^{(k,l)}$ and the fact that $a_{2i+1}^{(k,l)} = a_{2i+1}^{(l,k)}$, solving this recurrence gives Theorem 1.

To simplify notation in what follows, let $a_n = a_n^{(k,l)}$. To derive the recurrence (25), define a function

$$\Upsilon_n : \; L_{n-1}^{(k,l)} \times \mathbb{N} \;\; \rightarrow \;\; L_n^{(k,l)}$$

by $\Upsilon_n(\lambda, s) = \mu$, where

$$\begin{aligned}
\mu_1 \quad &\leftarrow \quad \left\lceil \frac{a_n \lambda_1}{a_{n-1}} \right\rceil + s \\
\mu_{2t} \quad &\leftarrow \quad \lambda_{2t-1}, \qquad 1 \leq t \leq n/2; \\
\mu_{2t+1} \quad &\leftarrow \quad \begin{cases} \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}} \right\rceil + \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t}, & 1 \leq t < (n-1)/2 \\ \\ \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t}, & t = (n-1)/2. \end{cases}
\end{aligned}$$

The key is to use the properties of the (k,l)-sequence to prove that $\mu \in L_n^{(k,l)}$, that Υ_n is a bijection, and that

$$|\mu|_e \quad = \quad |\lambda|_o; \tag{26}$$

$$|\mu|_o \quad = \quad \begin{cases} l|\lambda|_o - |\lambda|_e + s & \text{if } n \text{ is even} \\ k|\lambda|_o - |\lambda|_e + s & \text{if } n \text{ is odd.} \end{cases} \tag{27}$$

For then this implies that when n is even:

$$
\begin{aligned}
L_n^{(k,l)}(x,y) \triangleq \sum_{\mu \in L_n^{(k,l)}} x^{|\mu|_o} y^{|\mu|_e} &= \sum_{\lambda \in L_{n-1}^{(k,l)}} \sum_{s=0}^{\infty} x^{l|\lambda|_o - |\lambda|_e + s} y^{|\lambda|_o} \\
&= \frac{1}{1-x} \sum_{\lambda \in L_{n-1}^{(k,l)}} (x^l y)^{|\lambda|_o} (1/x)^{|\lambda|_e} = \frac{L_{n-1}^{(k,l)}(x^l y, x^{-1})}{1-x},
\end{aligned}
$$

giving the even case of recurrence (25) and the case for odd n is similar.

It remains to prove that Υ_n is a bijection satisying (26) and (27). First observe that since a is a (k,l)-sequence, for any $m \geq 0$

$$
\left\lceil \frac{a_{i+1}}{a_i} m \right\rceil + \left\lfloor \frac{a_{i-1}}{a_i} m \right\rfloor = \begin{cases} km & \text{if } i \text{ even} \\ lm & \text{if } i \text{ odd.} \end{cases}
$$

Thus, when n is even,

$$
|\mu|_o = \mu_1 + \mu_3 + \ldots = s + l(\lambda_1 + \lambda_3 + \ldots) - (\lambda_2 + \lambda_4 + \ldots) = l|\lambda|_o - |\lambda|_e + s,
$$

and similarly, for n odd, proving (27). Condition (26) is easy to check.

To show that $\mu \in L_n^{(k,l)}$, note that consecutive parts $\lambda_{2t-1}, \lambda_{2t}, \lambda_{2t+1}$ in λ, map to the consecutive parts of μ:

$$
\begin{aligned}
\mu_{2t} &= \lambda_{2t-1}, \\
\mu_{2t+1} &= \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}} \right\rceil + \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t}, \\
\mu_{2t+2} &= \lambda_{2t+1}.
\end{aligned}
$$

We need to show that

$$
\mu_{2t} \geq \frac{a_{n-2t+1}}{a_{n-2t}} \mu_{2t+1}; \quad \mu_{2t+1} \geq \frac{a_{n-2t}}{a_{n-2t-1}} \mu_{2t+2}.
$$

As $\lambda \in L_{n-1}^{(k,l)}$,

$$
\lambda_{2t-1} \geq \frac{a_{n-2t+1}}{a_{n-2t}} \lambda_{2t}; \quad \lambda_{2t} \geq \frac{a_{n-2t}}{a_{n-2t-1}} \lambda_{2t+1},
$$

so

$$
\mu_{2t+1} \geq \left\lceil \frac{a_{n-2t}\lambda_{2t+1}}{a_{n-2t-1}} \right\rceil = \left\lceil \frac{a_{n-2t}\mu_{2t+2}}{a_{n-2t-1}} \right\rceil
$$

and

$$
\mu_{2t+1} \leq \left\lfloor \frac{a_{n-2t}\lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor = \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor.
$$

Note that this did not require that the sequence $\{a_n\}$ be nondecreasing.

Finally, (λ, s) can be recovered from μ by:

$$
\begin{aligned}
s &\leftarrow \mu_1 - \left\lceil \frac{a_n \mu_2}{a_{n-1}} \right\rceil \\
\lambda_{2t-1} &\leftarrow \mu_{2t}, \quad 1 \leq t \leq n/2; \\
\lambda_{2t} &\leftarrow \begin{cases} \left\lceil \frac{a_{n-2t}\mu_{2t+2}}{a_{n-2t-1}} \right\rceil + \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor - \mu_{2t+1}, & 1 \leq t < (n-1)/2 \\ \left\lfloor \frac{a_{n-2t}\mu_{2t}}{a_{n-2t+1}} \right\rfloor - \mu_{2t+1}, & t = (n-1)/2. \end{cases}
\end{aligned}
$$

9 Appendix 2: Bijective proof of Theorem 2

Let $s_1, s_2, \ldots, s_k$ be a sequence of positive integers satisfying the condition $s_i = 1$ or $s_{i+1} = 1$ for $1 \le i \le k-1$. Recall that $b_1 = 1$ and $b_i = s_1 + \ldots + s_i$ for $i > 0$.

In the numerator of $F(q)$ in Theorem 2, write

$$1 + q^{b_i}([s_{i+1}]_q - 1) = q^{b_i+1} + q^{b_i+2} + \ldots + q^{b_i+s_{i+1}-1} = q^{b_i+1} + q^{b_i+2} + \ldots + q^{b_{i+1}-1}.$$

So, each positive integer in the set $\{b_1\} \cup \{b_2, b_2+1, b_2+3, \ldots, b_k\}$ occurs exactly once in $F(q)$ as an exponent of q, either in the numerator or the denominator. We can interpret $F(q)$ as the generating function for the set of partitions of an integer into parts from the set $\{1, b_2, b_2+1, \ldots, b_k\}$ in which parts in the set $\cup_{i=2}^{k-1}\{b_i+1, b_i+2, \ldots, b_{i+1}-1\}$ can occur at most once.

To prove Theorem 2, we give a weight-preserving bijection from the set of sequences $\lambda = \lambda_1, \ldots, \lambda_k$ satisfying

$$\frac{\lambda_1}{s_1} \ge \frac{\lambda_2}{s_2} \ge \cdots \ge \frac{\lambda_k}{s_k} \ge 0$$

to the set of pairs of partitions (μ, ν), where μ is a partition into parts in $\{b_1, \ldots, b_k\}$, and where ν is a partition into distinct parts from $\cup_{i=2}^{k-1}\{b_i+1, b_i+2, \ldots, b_{i+1}-1\}$.

```
Given λ, construct (μ, ν) as follows:
For i from k down to 2 do
        While λ_i/s_i ≥ 1 do
                For j from 1 to i do
                        λ_j ← λ_j − s_j
                μ ← μ ∪ b_i
        If λ_i > 0 then
                For j from 1 to i − 1 do
                        λ_j ← λ_j − s_j
                ν ← ν ∪ (b_{i−1} + λ_i)
μ ← μ ∪ b_1^{λ_1}
```

Inside the main loop, if $\lambda_i \ge s_i$ and

$$\frac{\lambda_1}{s_1} \ge \frac{\lambda_2}{s_2} \ge \cdots \ge \frac{\lambda_i}{s_i} \ge 0$$

then

$$\frac{\lambda_1 - s_1}{s_1} \ge \frac{\lambda_2 - s_2}{s_2} \ge \cdots \ge \frac{\lambda_i - s_i}{s_i} \ge 0$$

and another part $s_1 + \ldots + s_i = b_i$ is added to μ. If $0 < \lambda_i < s_i$, then $s_i \ge 2$. By definition of s, then $s_{i-1} = 1$, so $\lambda_{i-1} \ge s_{i-1}$ and

$$\frac{\lambda_1 - s_1}{s_1} \ge \frac{\lambda_2 - s_2}{s_2} \ge \cdots \ge \frac{\lambda_i - s_{i-1}}{s_{i-1}} \ge 0$$

and, for the first and only time, part $s_1 + \ldots + s_{i-1} + \lambda_i = b_{i-1} + \lambda_i < b_i$ is added to ν.

The reverse is straightforward. $\square$

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